

## Proof of the EXPLICIT NON-RECURSIVE FORMULA.

Here is the proof of the explicit, non-recursive solution of general sum recurrences of the form

$$a_1 = r_{1,1} \tag{1}$$

$$a_n = \sum_{m=1}^{n-1} r_{n,m} a_m \tag{2}$$

for  $a_n$ . The formula was first derived informally via pattern recognition, but that is not a rigorous proof that it is really correct.

The explicit solution of this recurrence can be given in terms of binary bit-manipulation functions. Namely, let, for a nonnegative integer  $N$ ,

$\text{n1bits}(N)$ : count of the 1 bits in  $N$ .  $\text{nth1idx}(N, n)$ : position of the  $n$ th 1 bit in  $N$ , with the LSB indicated as position 1. For the purposes of this formula, when  $n = 0$ , this is taken to equal 1.

These two functions are related by

$$N = \sum_{n=1}^{\text{n1bits}(N)} 2^{\text{nth1idx}(N, n) - 1} \tag{3}$$

Now, we can state the explicit, non-recursive solution as

$$a_n = \sum_{j=1}^{2^{n-2} \text{n1bits}(2^{n-1} + 2j - 1)} \prod_{k=1}^j r_{\text{nth1idx}(2^{n-1} + 2j - 1, k), \text{nth1idx}(2j - 1, k - 1)} \tag{4}$$

Before stating the proof, we first prove some properties of the binary bit functions that will be useful.

### Properties of $\text{nth1idx}$ .

1.  $\text{nth1idx}(2N, n) = \text{nth1idx}(N, n) + 1$
2. For even  $N$ ,  $\text{nth1idx}(N + 1, n + 1) = \text{nth1idx}(N, n)$ .

**Proof.** For property 1, note that multiplying by 2 is the same as shifting the bits left by 1. Thus, every 1 bit is bumped up one place in significance, while the number of 1 bits remains unchanged. Hence, the index of the  $n$ th 1 bit increases by 1.

For property 2, note that the positions of all the 1 bits remain unchanged, but a new 1 bit is gained at the LSB. Thus, the 1st one bit is now the LSB, 1, while the 2nd, 3rd, etc. now count into the remaining bits, thus correspond to the 1st, 2nd, etc. bits in the original number. QED.

With this in hand, we can now provide the proof of the explicit formula.

**Theorem.** Formula (4) is the solution of recurrence (2).

**Beginning of proof.** Before proving the main result, we will first prove the following lemmas. Here, we will also consider  $r_{n,m}$  in the abstract, that is,  $r_{n,m}$  are all distinct, i.e. for two different pairs of indices  $n_1$  and  $m_1$ ,  $n_2$  and  $m_2$ , we treat it as though  $r_{n_1,m_1} \neq r_{n_2,m_2}$ , regardless of the equality of the actual values of  $r_{n,m}$  in a specific instance, and a product or sum of two different-index these cannot collapse in any way, i.e.  $r_{n_1,m_1}r_{n_2,m_2}$  is a unique combination (up to commutativity, etc.). i.e. we are manipulating purely formal expressions in  $r_{n,m}$  here.

It is obvious that  $a_n$  is a sum of products of the  $r_{n,m}$ .

**Lemma 1.** Every product in  $a_n$  has no repeated  $r_{n,m}$ , i.e. all appearing  $r_{n,m}$  have exponent 1.

**Proof.** Looking at recurrence (4), we see that for all  $a_i$  with  $i < n$ , there can exist no  $r_{n,?}$ , since  $r_{n,?}$  is only involved in the sums when  $i = n$ . This means that  $r_{n,m}$  only appears in  $a_n$  and later. In  $a_n$ , it only multiplies  $a_m$ , which has no instances of it, since  $m < n$ .  $r_{n,n}$  does not appear, due to the sum bounds. In succeeding  $a_{n+k}$ ,  $k > 0$ , it will only get multiplied by  $r_{n+k,?}$ . Since  $k > 0$ , this also cannot be  $r_{n,m}$ . Thus, the exponent of any appearing  $r_{n,m}$  is 1. Qed.

**Lemma 2.** The products added together in each sum  $a_n$  are all distinct, i.e. they have no constant multiplying coefficients other than 1 or no terms collapse.

**Proof.** Suppose that at some sum  $a_N$ , there is a first appearance of such an incident. As a first appearance, none of the  $a_m$  in the sum have within them any nondistinct products. This then implies that there exist  $r_{N,m_1}a_{m_1}$  and  $r_{N,m_2}a_{m_2}$ , with  $m_1 \neq m_2$  such that there is a shared product common to both. Consider this product. Its existence means it is equal to a product from  $a_{m_1}$  times  $r_{N,m_1}$ , and also to another product from  $a_{m_2}$  times  $r_{N,m_2}$ . The product from  $a_{m_1}$  cannot equal that from  $a_{m_2}$ , because  $r_{N,m_1}$  would then have to equal  $r_{N,m_2}$ , which is impossible since  $m_1 \neq m_2$  (remember, we're manipulating formal expressions). If it is not equal, then that would mean one or the other of  $r_{N,m_1}$  or  $r_{N,m_2}$  would have to cancel something in the product to leave the other. But no  $r_{N,?}$  appear in either  $a_{m_1}$  or  $a_{m_2}$ , per the

argument presented in the proof of Lemma 1, and there are no quotients, either, since division is not present in any of the expressions (3) and (4). Hence, there could be no shared products in  $r_{N,m_1}a_{m_1}$  and  $r_{N,m_2}a_{m_2}$ . Qed.

**Lemma 3.** The number of products in each sum  $a_n$  is  $2^{n-2}$ , for  $n > 1$ .

**Proof.** Since Lemma 2 says that each product comprising the  $a_n$  is distinct (i.e. pattern-distinct in the sense of the formal expressions), replacing each product by 1 will count the number of products. This can be done by setting  $r_{n,m}$  to 1, so the products, all composed of  $r_{n,m}$ , acquire value 1, thus their sum is equal to their number. Then, we obtain for the number of products  $NP_n$  in  $a_n$ ,

$$NP_1 = 1 \tag{5}$$

$$NP_n = \sum_{m=1}^{n-1} NP_m \tag{6}$$

The equation (6) for  $NP_n$  is a sum equation, which can be turned via the forward difference operator into a difference equation:

$$\Delta NP_n = \left( \sum_{m=1}^n NP_m \right) - \left( \sum_{m=1}^{n-1} NP_m \right) \tag{7}$$

$$= NP_n, n > 1 \tag{8}$$

This difference equation has a general solution of  $NP_n = K2^n + L$ . Using the initial condition that at  $n = 2$ , this should be 1, and that at  $n = 3$ , this should be 2 (which can both be verified by brute application of (5) and (6)), we get

$$4K + L = 1 \tag{9}$$

$$8K + L = 2 \tag{10}$$

This implies  $L = 1 - 4K$  and  $L = 2 - 8K$ . Substituting  $L = 1 - 4K$  into the second equation gives  $8K + 1 - 4K = 2$ , or  $4K + 1 = 2$ . This means  $4K = 1$ ,  $K = \frac{1}{4}$ , thus  $L = 1 - \frac{4}{4} = 1 - 1 = 0$ , and so  $NP_n = \frac{1}{4}2^n = 2^{n-2}$ ,  $n > 1$ . QED.

**Main proof.** We now know that each sum  $a_n$  is a sum of  $2^{n-2}$  products  $r_{n,m}$ , for  $n > 1$ . Now suppose that all the products in the  $a_n$  are lined up in an infinite sequence, with successive blocks of products representing those in  $a_n$ . We do not specify the order within the blocks: as we proceed and assign values to the sequence members, that will be defined. For now, we just write

$$a_n = \sum_{j=1}^{2^{n-2}} p_{j+\text{orig}(n)} \quad (11)$$

where  $\text{orig}(n)$  is the initial (or origin) product in this term, and  $p_j$  is the infinite sequence of all the products in the various terms. To determine  $\text{orig}(n)$ , we note each term has  $2^{n-2}$  products. So, after  $n$  terms, we have gone past  $1 + 2^{2-2} + 2^{3-2} + \dots + 2^{n-2} = 1 + \sum_{j=2}^n 2^{j-2} = 1 + \sum_{j=0}^{n-2} 2^j = 1 + (2^{n-1} - 1) = 2^{n-1}$  products. After  $n - 1$  terms, we have gone past  $2^{n-2}$  products, thus  $\text{orig}(n) = 2^{n-2}$ , hence

$$a_n = \sum_{j=1}^{2^{n-2}} p_{j+2^{n-2}}, n > 1. \quad (12)$$

Now, consider the recurrence formula (4) in terms of the sequence of products. We expand  $a_m$  as

$$a_m = \sum_{k=1}^{2^{m-2}} p_{k+2^{m-2}}, m > 1, \quad (13)$$

thus,

$$a_n = \sum_{m=1}^{n-1} r_{n,m} a_m \quad (14)$$

$$= r_{n,1} r_{1,1} + \sum_{m=2}^{n-1} r_{n,m} a_m \quad (15)$$

$$= r_{n,1} r_{1,1} + \sum_{m=2}^{n-1} r_{n,m} \left( \sum_{k=1}^{2^{m-2}} p_{k+2^{m-2}} \right) \quad (16)$$

$$= r_{n,1} r_{1,1} + \sum_{m=2}^{n-1} \sum_{k=1}^{2^{m-2}} r_{n,m} p_{k+2^{m-2}}. \quad (17)$$

We seek to convert the sum on the right to a single sum over the  $p$ . If we consider the index of  $p$  as a single index  $j$  of summation, we get  $j = 2^{m-2} + k$ , with  $k \leq 2^{m-2}$ . We now want to invert this, to recover  $m$  and  $k$  from  $j$ . Consider  $j - 1 = 2^{m-2} + (k - 1)$ . Now  $0 \leq k - 1 < 2^{m-2}$ . Note then that  $\lfloor \log_2(j - 1) \rfloor = m - 2$ , since the addition of  $k - 1 < 2^{m-2}$  does not raise it over the next power of 2. Thus,  $m = \lfloor \log_2(j - 1) \rfloor + 2$ . Then,  $k - 1 = j - 1 - 2^{m-2}$  for that  $m$ . But we only need  $m$ , and we can write

$$a_n = r_{n,1}r_{1,1} + \sum_{j=2}^{2^{n-2}} r_{n, \lfloor \log_2(j-1) \rfloor + 2} p_j. \quad (18)$$

The right side defines the product-sequence members for this term, i.e.  $p_{j+2^{n-2}}$ , giving a new recurrence for individual products:

$$p_{1+2^{n-2}} = r_{n,1}r_{1,1}p_{j+2^{n-2}} = r_{n, \lfloor \log_2(j-1) \rfloor + 2} p_j, 1 < j \leq 2^{n-2}. \quad (19)$$

Noting that  $p_1 = r_{1,1}$ , we have

$$p_{1+2^{n-2}} = r_{n,1}p_1 \quad (20)$$

This means that if we take

$$F(j) = \begin{cases} \lfloor \log_2(j - 1) \rfloor + 2, & \text{if } j \neq 1 \\ 1, & \text{if } j = 1 \end{cases} \quad (21)$$

we get

$$p_{j+2^{n-2}} = r_{n, F(j)} p_j, 1 \leq j \leq 2^{n-2}. \quad (22)$$

Now, the presence of the binary counting is clear. Consider the expansion of  $j - 1$ , so that  $(j - 1) + 1 = j$ . This gives

$$j - 1 = \sum_{k=0}^{\lfloor \log_2(j-1) \rfloor} b_k 2^k \quad (23)$$

where  $b_k$  are the bits. Alternatively, we can express this as

$$j - 1 = \sum_{k=1}^{\text{n1bits}(j-1)} 2^{\text{nth1idx}(j-1, k) - 1}. \quad (24)$$

Then,

$$j = 1 + \sum_{k=1}^{\text{n1bits}(j-1)} 2^{\text{nth1idx}(j-1,k)-1}. \quad (25)$$

Note that each partial sum is less than the next term added, i.e. if

$$j_s = 1 + \sum_{k=1}^s 2^{\text{nth1idx}(j-1,k)-1} \quad (26)$$

then  $j_s <= 2^{\text{nth1idx}(j-1,s+1)-1}$ . This can be seen by noting that  $\text{nth1idx}(j-1, k)$  is always increasing with respect to  $k$  as  $k$  goes from 1 to  $\text{n1bits}(j-1)$ , per its definition (incrementing  $k$  yields the index of the next 1 bit *after* the  $k$ th, so it *must* give a higher number.), thus  $\lfloor \log_2(j_s) \rfloor = \lfloor \log_2(< \text{its largest term } >) \rfloor = \text{nth1idx}(j-1, s) - 1 < \text{nth1idx}(j-1, s+1) - 1$ . Finally, for even  $j$ ,  $j-1$  has a 1 bit as its LSB, thus  $j_1 = 1 + 1$  and  $1 \leq 1$ . Thus, we can apply the recurrence repeatedly to obtain the product  $p_j$ :

$$p_j = p_1 \prod_{k=1}^{\text{n1bits}(j-1)} r_{\text{nth1idx}(j-1,k)+1, F(1+\sum_{s=1}^{k-1} 2^{\text{nth1idx}(j-1,k)-1})} \quad (27)$$

$$= r_{1,1} \prod_{k=1}^{\text{n1bits}(j-1)} r_{\text{nth1idx}(j-1,k)+1, F(1+\sum_{s=1}^{k-1} 2^{\text{nth1idx}(j-1,k)-1})}. \quad (28)$$

Now, consider  $F\left(1 + \sum_{s=1}^{k-1} 2^{\text{nth1idx}(j-1,s)-1}\right)$ . This is 1 when  $k-1 = 0$ , otherwise, it is  $\lfloor \log_2\left(\sum_{s=1}^{k-1} 2^{\text{nth1idx}(j-1,s)-1}\right) \rfloor = \text{nth1idx}(j-1, k-1) + 1$ . Then,

$$p_j = r_{1,1} r_{\text{nth1idx}(j-1,1)+1,1} \prod_{k=2}^{\text{n1bits}(j-1)} r_{\text{nth1idx}(j-1,k)+1, \text{nth1idx}(j-1,k-1)+1}. \quad (29)$$

Now,

$$a_n = \sum_{j=1}^{2^{n-2}} p_{j+2^{n-2}} \quad (30)$$

$$= r_{1,1} + \sum_{j=2}^{2^{n-2}} r_{1,1} r_{\text{nth1idx}(2^{n-2}+j-1,1),1} \prod_{k=2}^{\text{n1bits}(2^{n-2}+j-1)} r_{\text{nth1idx}(2^{n-2}+j-1,k)+1, \text{nth1idx}(2^{n-2}+j-1,k-1)+1}. \quad (31)$$

But is this equivalent to our formula from the pattern recognition? It turns out, it is entirely equivalent. By property 1 of `nth1idx`,

$$r_{\text{nth1idx}(2^{n-2}+j-1,k)+1, \text{nth1idx}(2^{n-2}+j-1,k-1)+1} = r_{\text{nth1idx}(2^{n-1}+2j-2,k), \text{nth1idx}(2^{n-1}+2j-2,k-1)}. \quad (32)$$

We seek to extend the product to subsume the two terms pulled outside. To bring back in the 2nd term, we note that if we take `nth1idx(N, 0) = 1` as before, it comes back in:

$$r_{1,1} r_{\text{nth1idx}(2^{n-2}+j-1,1),1} \prod_{k=2}^{\text{n1bits}(2^{n-2}+j-1)} r_{\text{nth1idx}(2^{n-1}+2j-2,k), \text{nth1idx}(2^{n-1}+2j-2,k-1)} = \quad (33)$$

$$r_{1,1} \prod_{k=1}^{\text{n1bits}(2^{n-2}+j-1)} r_{\text{nth1idx}(2^{n-1}+2j-2,k), \text{nth1idx}(2^{n-1}+2j-2,k-1)} = \quad (34)$$

$$r_{1,1} \prod_{k=1}^{\text{n1bits}(2^{n-1}+2j-2)} r_{\text{nth1idx}(2^{n-1}+2j-2,k), \text{nth1idx}(2^{n-1}+2j-2,k-1)}, n > 1 \quad (35)$$

How can we subsume  $r_{1,1}$ ? Note that  $2^{n-1} + 2j - 2$  is always even, so its LSB is always 0. If we add 1 to it, then it will equal  $2^{n-1} + 2j - 1$ , which is always odd, with an LSB of 1. The index of the first 1 bit of it will then always be 1, while the indices of the 1 bits beyond will regurgitate the sequence obtained from  $2^{n-1} + 2j - 2$  starting at  $k = 1$ . Then, using the second property of `nth1idx`,

$$r_{\text{nthlidx}(2^{n-1}+2j-2,k),\text{nthlidx}(2^{n-1}+2j-2,k-1)} = r_{\text{nthlidx}(2^{n-1}+2j-1,k+1),\text{nthlidx}(2^{n-1}+2j-1,k)}, k > 0. \quad (36)$$

At  $k = 0$ , the second formula yields  $r_{1,1}$ , thus it subsumes that term in, and so

$$r_{1,1} \prod_{k=1}^{\text{n1bits}(2^{n-1}+2j-2)} r_{\text{nthlidx}(2^{n-1}+2j-2,k),\text{nthlidx}(2^{n-1}+2j-2,k-1)} = \quad (37)$$

$$\prod_{k=0}^{\text{n1bits}(2^{n-1}+2j-2)} r_{\text{nthlidx}(2^{n-1}+2j-1,k+1),\text{nthlidx}(2^{n-1}+2j-1,k)} = \quad (38)$$

$$\prod_{k=1}^{\text{n1bits}(2^{n-1}+2j-2)+1} r_{\text{nthlidx}(2^{n-1}+2j-1,k),\text{nthlidx}(2^{n-1}+2j-1,k-1)} = \quad (39)$$

$$\prod_{k=1}^{\text{n1bits}(2^{n-1}+2j-1)} r_{\text{nthlidx}(2^{n-1}+2j-1,k),\text{nthlidx}(2^{n-1}+2j-1,k-1)}. \quad (40)$$

But wait... the second index in our pattern-recognition formula was  $\text{nthlidx}(2j-1, k-1)$ . Can we do that here? Yes. Note that  $k-1$  goes only up to  $\text{n1bits}(2^{n-1}+2j-1)$ , so the highest 1 bit the one supplied by  $2^{n-1}$  is never counted. Thus we can drop it off, and we get

$$\prod_{k=1}^{\text{n1bits}(2^{n-1}+2j-1)} r_{\text{nthlidx}(2^{n-1}+2j-1,k),\text{nthlidx}(2^{n-1}+2j-1,k-1)} = \quad (41)$$

$$\prod_{k=1}^{\text{n1bits}(2^{n-1}+2j-1)} r_{\text{nthlidx}(2^{n-1}+2j-1,k),\text{nthlidx}(2j-1,k-1)}. \quad (42)$$

Then,

$$a_n = \sum_{n=1}^{2^{n-2} \text{n1bits}(2^{n-1}+2j-1)} \prod_{k=1}^{\text{n1bits}(2^{n-1}+2j-1)} r_{\text{nthlidx}(2^{n-1}+2j-1,k),\text{nthlidx}(2j-1,k-1)}, \quad (43)$$

which is identical to the pattern-recognition formula, and the proof is complete. Q.E.D.