Tetration Reference

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July 10, 2008

Contents

Chapter 1

Introduction

This Reference is meant to provide an in-depth, detailed understanding about tetration and its related topics.

The Reference need not be understandable by everyone, and may include advanced topics.

1.1 Classification

1.2 Exponentiation

Exponentiation Exponentiation is a binary operation on $\mathbb{C}$ defined by:

$$
\boxed{\quad b^1 := b \quad \Big| \quad b^a := b b^{a-1} \quad \Big| \quad b^a := e^{\ln(b)a} \quad \Big| \quad e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \Big| \quad \ln(x) := \sum_{k=1}^{\infty} \frac{(1-x)^k}{-k} \quad}
$$

Table of Values. Exponentiation near singularities can be summarized by:

	$a = -\infty$	$\Re(a) < 0$	$\Re(a) = 0$	$\Re(a) > 0$	$a = \infty$
$b = 0$	∞	∞	$\frac{0}{0}$	0	0
$0 < |b| < 1$	∞	b^a	b^a	b^a	0
$|b| = 1$	$\frac{0}{0}$	$e^{i\theta}$	$\left\{\begin{array}{ll} 1 & \text{if } a = 0, \\ 1 & \text{if } b = 1, \\ e^{i\theta} & \text{otherwise} \end{array}\right\}$	$e^{i\theta}$	$\frac{0}{0}$
$|b| > 1$	0	b^a	b^a	b^a	∞
$|b| = \infty$	0	0	$\frac{0}{0}$	∞	∞

where ∞ is complex infinity, and $\frac{0}{0}$ is an indeterminate value. The expression $\boldsymbol{e}^{i\theta}$ is meant to indicate that the result has magnitude 1, with the angle or argument: $\theta = a\operatorname{Arg}(b)$.

Chapter 2

Hyper-operations

Hyper-operations are a generalization of the normal sequence of binary operations that are ubiquitous in mathematics and science. The sequence of binary operations: addition, multiplication, and exponentiation, continues beyond exponentiation [46], but this continuation is not unique. The term **hyper-operation** refers to any binary operation on $\mathbb{N}$ defined recursively in terms of the operations that come before it. There are quite a few continuations of this sequence, and any set that includes these operations can be called a hyper-operation set. It is customary to assign addition to hyper-1 and multiplication to hyper-2, but a different **operation succession** can result in different operations beyond exponentiation.

The **standard hyper-operations** use

$$b\,\boxed{n}\,c = b\,\boxed{n-1}\,(b\,\boxed{n}\,(c-1)) \tag{2.1}$$

as the definition of operation succession, and the operations defined from this form what is called *the* **hyper-operation sequence**. A common name for this sequence is the Ackermann function, specifically the three argument version. The Ackermann function can either refer to the 2 or 3 argument versions, defined as

$$A_2(n,m) = (2\,\boxed{n}\,(m+3)) - 3 \tag{2.2}$$

$$A_3(p,n,m) = p\,\boxed{n}\,m \tag{2.3}$$

respectively. Since the three argument Ackermann function is identical with the standard hyper-operations, we will use the later for both notation and terminology. Another concept is the more general Grzegorczyk or Grzegorczyk-Wainer hierarchy.

The nth hyper-operation is called the **hyper-n-operation**[1] or **hyper-n** for short, so hyper-1 is addition, hyper-2 is multiplication, hyper-3 is exponentiation, and so on.

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[1] All of these terms can be generalized from hyper-N-*term* to hyper-*term* to refer to all ranks.

In 1947, Robert L. Goodstein [50] coined terms for hyper-operations above hyper-3. His terminology borrows from the Greek number prefixes, so hyper-4 is **tetration**[2], hyper-5 is **pentation**, hyper-6 is **hexation**, and so on. In general the *hyper-* prefix denotes all ranks, and the *super-* prefix denotes rank 4.

Here is an overview of the terminology used with common hyper-operations:

N	2	3	4
hyper-N-operation	multiplication	exponentiation	tetration
hyper-N-power	multiple	power	tetrate
hyper-N-exponential	multiple	exponential	tetrational
hyper-N-root	division	root	super-root
hyper-N-logarithm	division	logarithm	super-logarithm
hyper-N-base	factor	base	base
hyper-N-exponent	factor	exponent	height
hyper-N-nest	n-ary product	nested exponential	nested tetrational

Although hyper-operations in general are only defined over $\mathbb{N} \times \mathbb{N}$, the first three are defined over $\mathbb{C} \times \mathbb{C}$. There are several methods for extending the others' domains to complex numbers, which will be discussed later.

2.1 Standard notations

There are two major kinds of notations in common use for hyper-operations, one uses an arrow ($\uparrow^n$) and the other uses a box ($\boxed{n}$). Each notation has at least two variants in common use. The box notation coined by Rubstov and Romerio [84] defines addition as $b\boxed{1}c = b + c$. The superscript notation coined by Munafo [80] is identical to box notation, defined as $b^{(n)}c = b\boxed{n}c$. The arrow notation coined by Knuth defined the arrow as exponentiation, so is offset from the other notations by two, defined as $b\uparrow^{n-2}c = b\boxed{n}c$. The arrow notation coined by Bromer is defined as $b\uparrow^{n-3}c = b\boxed{n}c$ being offset by three. Next, we discuss these two versions of arrow notation.

2.1.1 Box notations

Rubstov and Romerio coined box notation, which is similar to Munafo's superscript notation ($b^{(n)}c$) since it assigns the same operations to the same values of n. Box

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[2]Two other terms for tetration are hyper-4 and iterated exponentials. The term *nth tetrate* is rather new; previous terms were *tower*, *hyperpower*, and *power tower*. The term *tetrational* was coined by Holmes as *antitetrational*, used in his draft proposal for *Composite Arithmetic*.

notation allows the notation of both hyper-logarithms and hyper-roots as well, an advantage that has motivated the use of box notation in this work.

$$a = b\boxed{n}c := \begin{cases} b + c & \text{if } n = 1, \\ b & \text{if } n > 1 \text{ and } c = 1, \\ b\boxed{n-1}(b\boxed{n}(c-1)) & \text{otherwise.} \end{cases} \qquad \begin{aligned} b &= {}^{c}_{n}\!\sqrt{a} \\ c &= {}^{n}_{b}\!\sqrt{a} \end{aligned}$$

The box notation for hyper-roots $\left({}^{c}_{n}\!\sqrt{a}\right)$ resembles a lowercase "r" for root, and the box notation for hyper-logarithms $\left({}^{n}_{b}\!\sqrt{a}\right)$ resembles an uppercase "L" for log. These notations provide an excellent mnemonic for remembering which is which, and helps with learning the new notation as well.

2.1.2 Arrow notations

There are two variants of arrow notation, Knuth's and Bromer's, each of which has some logic behind them. Knuth's up-arrow notation is defined by:

$$a = b{\uparrow}^{n}c := \begin{cases} b^{c} & \text{if } n = 1, \\ b & \text{if } n > 1 \text{ and } c = 1, \\ b{\uparrow}^{n-1}(b{\uparrow}^{n}(c-1)) & \text{otherwise.} \end{cases}$$

which means $b{\uparrow}c = b^{c}$ (exponentiation) and $b{\uparrow}{\uparrow}c = {}^{c}b$ (tetration). Knuth's up-arrow notation is the primary notation used for hyper-operations, also used in [100].

Bromer's [22] arrow notation, also used by Müller [78], is more general than Knuth's up-arrow notation. Instead of being restricted to the right-associative iteration of succession $(x+1)$, it allows either right-associative or left-associative iteration of *any binary operation*. The Bromer-Müller arrow notation is defined as:

$$b{\uparrow}Ac := bA(bA \cdots A(bAb)) \text{ with } c \text{ } b\text{'s} \tag{2.4}$$
$$b{\downarrow}Ac := ((bAb)A \cdots Ab)Ab \text{ with } c \text{ } b\text{'s} \tag{2.5}$$

where $b{\uparrow}c = {}^{c}b$ and $b{\downarrow}c = b^{b^{(c-1)}}$ in Bromer's and Müller's original usage, whereas $b{\uparrow}c = b{\downarrow}c = b^{c}$ to simulate Knuth's notation. Since this allows for two notations for every successive operation, we will use $b{\uparrow}c = b^{c}$, $b{\uparrow}{\uparrow}c = {}^{c}b$ and $b{\downarrow}{\downarrow}c = b^{b^{(c-1)}}$ as the convention of keeping the last two arrows pointing in the same direction.

N	Term	Notation	N	Term	Notation
L	hyper-L	$b{\downarrow}{\downarrow}a = b^{b^{(a-1)}}$	R	tetration	$b{\uparrow}{\uparrow}a = b\boxed{4}a = {}^{a}b$
LR	hyper-LR	$b{\downarrow}{\uparrow}{\uparrow}a$	RR	pentation	$b{\uparrow}{\uparrow}{\uparrow}a = b\boxed{5}a$
LL	hyper-LL	$b{\downarrow}{\downarrow}{\downarrow}a$	RL	hyper-RL	$b{\uparrow}{\downarrow}{\downarrow}a$

2.1.3 Conway's chain notation

Conway's chained-arrow notation is one of two generalizations of hyper-operations from a ternary operation to an n-ary operation. It is easy to define this notation in terms of iteration notation $f^{on}(x)$, because it embeds iteration. In the following definition $\mathbf{x} = a \to \cdots \to z$ stands for a partial Conway chain of any length.

$$b \to c := b^c \tag{2.6}$$

$$\mathbf{x} \to 1 := \mathbf{x} \tag{2.7}$$

$$\mathbf{x} \to b \to c := CA_{(\mathbf{x};c)}(b) \tag{2.8}$$

$$CA_{(\mathbf{x};c+1)}(b) := CA_{(\mathbf{x};c)}^{ob}(1) \tag{2.9}$$

as you can see, the second-to-last element plays the role of the iterator.

The 3-element Conway chains are equivalent to hyper-operations:

$$b \to c \to n = b \uparrow^n c \tag{2.10}$$

2.1.4 Bowers' array notation

Jonathan Bowers' array notations use arrays to store small positive integers that evaluate to a single number. In Bowers' description [20] of his array notations, he makes little distinction between an *array* and the *evaluation* of that array. To make this distinction more clear, parentheses $(\mathbf{a})$ will be used for the array itself, and square brackets $[\mathbf{a}]$ will be used for evaluation. Also used is the convention: $\mathbf{a} = \{a_k\}_{k=1}^n$, meaning $\mathbf{a}$ is a sequence with n elements where the kth element is written a_k. In his original description of his array notation, he gives five rules. Due to the complexity of rules 4 and 5, they have been broken down as follows.

Let $rule4(\mathbf{a})$ be $\mathbf{a}$ with the following replacements:

$$
\begin{aligned}
a_k &\to a_1 \text{ for } 2 \le k \le (j-2) \\
a_{j-1} &\to [\mathbf{a}|_{a_2 \to (a_2-1)}] \\
a_j &\to a_j - 1
\end{aligned}
$$

where $a_i = 1$ for $3 \le i \le (j-1)$ and $a_j \ne 1$, (i.e., a_j is the first non-1 after a_3).

Let $rule5(\mathbf{a})$ be $\mathbf{a}$ with the following replacements:

$$
\begin{aligned}
a_2 &\to [\mathbf{a}|_{a_2 \to (a_2-1)}] \\
a_3 &\to a_3 - 1
\end{aligned}
$$

Using $rule4(\mathbf{a})$, $rule5(\mathbf{a})$, and the array $\mathbf{a} = \{a_k\}_{k=1}^n = (a_1, a_2, ..., a_n)$:

$$[\mathbf{a}] = \begin{cases} a_1 & \text{if } n = 1, \\ a_1 + a_2 & \text{if } n = 2, \\ [\{a_k\}_{k=1}^{n-1}] & \text{if } a_n = 1, \\ a_1 & \text{if } a_2 = 1, \\ [rule4(\mathbf{a})] & \text{if } a_3 = 1, \\ [rule5(\mathbf{a})] & \text{otherwise.} \end{cases} \tag{2.11}$$

is the evaluation of Bowers' original array notation.

The way this notation is defined, it is equivalent to the hyper-operation sequence for arrays of three elements. Using $\mathbf{a} = (x, y, n)$, the following relationship holds:

$$[b, \ c, \ n] = b\boxed{n}c \tag{2.12}$$

In Bowers' description, he mentions that Chris Bird proved that 4-element Bowers arrays are "comparible" to 4-element Conway chains, but what this means is unclear.

2.1.5   Steinhaus-Moser notation

Hugo Steinhaus coined a notation for large numbers that defined three elements: n inside a triangle means n^n, n inside a square means n inside n nested triangles, and n inside a circle means n inside n nested squares. Steinhaus's notation is also called **circle notation**, due to the fact that it uses a circle for nested squares. Leo Moser then extended this notation by defining n inside a p-gon (p sided polygon) to mean n inside n nested $(p-1)$-gons. Moser's extension of Steinhaus's notation has been called **polygon notation**. The two forms that are common to both notations are the triangle (SM_3) and the square (SM_4), which are defined by:

$$SM_3(n) := n^n \tag{2.13}$$
$$SM_p(n) := SM_{p-1}^{\circ n}(n) \tag{2.14}$$

where $SM_p^{\circ m}(n)$ represents an n inside m nested p-gons.

Three very popular large numbers that are defined using this notation are:

- **mega** := $SM_5(2)$

- **megiston** := $SM_5(10)$

- **Moser's number** := $SM_{SM_5(2)}(2)$

10

2.2 Standard definitions

Having the ability to calculate hyper-operations is not sufficient for manipulating expressions involving them. In order to do algebraic manipulations, we also need their inverse functions, and the ability to calculate them as well. All hyper-operations have two inverses, **hyper-roots** and **hyper-logarithms**. For rank 3, roots are "easier" to find than logarithms, as they can be defined as $\sqrt[c]{a} = a^{1/c}$, but for all ranks 4 and above, hyper-logarithms seem to be "easier" to find than hyper-roots.

2.2.1 Hyper-exponentials

A **hyper-exponential** is a univariate function defined as $f(x) = b\boxed{n}x$. In order to discuss hyper-exponential functions, it is good idea to extend them to continuous or differentiable functions defined on the real or complex numbers. Although no such an extension has been found, there is a generally accepted framework for discussing such extensions. There are two views of this framework:

Section-based A hyper-exponential is defined as a collection (a sheaf) of series expansions (sections) that all satisfy the operation succession.

Piece-based A hyper-exponential is piecewise-defined by the operation succession and is differentiable across the boundary of its critical piece.

and the results from both approaches are equivalent. So it makes little difference which approach is taken, since they can both produce differentiable extensions.

Given two **critical piece** functions defined over intervals of length one:

- $HC_{(b,d)}(c)$ defined over $0 \leq b \in \mathbb{R}$, $c_0 < c < (c_1 = c_0 + 1)$, $d_1 < d \in \mathbb{R}$, and

- $HD_{(b,d)}(c)$ defined over $0 \leq b \in \mathbb{R}$, $c \in \mathbb{R}$, $d_0 < d < (d_1 = d_0 + 1)$,

the standard hyper-operations may be extended to a ternary operation on $\mathbb{R}$ as:

$$
b\boxed{d}c = \begin{cases}
b\boxed{d+1}\left(\lfloor_b^{d+1}c + 1\right) & \text{if } 0 \leq d < d_0 \\
HD_{(b,d)}(c) & \text{if } d_0 \leq d \leq d_1 \\
\left.\begin{cases}
\lfloor_b^{d-1}\lfloor b\boxed{d}(c+1) & \text{if } c < c_0 \\
HC_{(b,d)}(c) & \text{if } c_0 \leq c \leq c_1 \\
b\boxed{d-1}(b\boxed{d}(c-1)) & \text{if } c > c_1
\end{cases}\right\} & \text{if } d > d_1
\end{cases}
$$

where the functions that best define $HC_{(b,d)}(c)$ and $HD_{(b,d)}(c)$ remain unknown. This definition effectively pushes all values within the domains of the critical pieces, thus reducing the domain of interpolation.

2.2.2 Hyper-logarithms

A **hyper-logarithm** is the inverse function of a hyper-exponential. The extension of hyper-exponentials to $\mathbb{R}$ requires the extension of hyper-logarithms to $\mathbb{R}$ as well.

Given the inverse functions of the **critical piece** functions above:

$$
{}_{b}^{d}\underline{a} =
\begin{cases}
b\,\boxed{d+1}\left({}_{b}^{d+1}\underline{a} - 1\right) & \text{if } 0 \le d < d_0 \\[4pt]
HD_{(b,d)}^{-1}(a) & \text{if } d_0 \le d \le d_1 \\[4pt]
\left\{
\begin{array}{ll}
\left({}_{b}^{d}\underline{b\,\boxed{d-1}\,a}\right) - 1 & \text{if } a < (b\,\boxed{d}\,c_0) \\[4pt]
HC_{(b,d)}^{-1}(a) & \text{if } (b\,\boxed{d}\,c_0) \le a \le (b\,\boxed{d}\,c_1) \\[4pt]
\left({}_{b}^{d}\underline{{}_{b}^{d-1}\,a}\right) + 1 & \text{if } a > (b\,\boxed{d}\,c_1)
\end{array}
\right\} & \text{if } d > d_1
\end{cases}
$$

which completes the framework for extension of hyper-operations to real numbers.

Here is an example of an extension for $c_0 = 0$ and $d_0 = 2$:

$$
HC_{(b,d)}(c) \approx b^c \tag{2.15}
$$

$$
HD_{(b,d)}(c) \approx bc \cos^2\left(\frac{d\pi}{2}\right) + b^c \sin^2\left(\frac{d\pi}{2}\right) \tag{2.16}
$$

where the square indicates a power, not an iterate.

2.2.3 Hyper-powers

A **hyper-power** is a univariate function defined as $f(x) = x\,\boxed{n}\,c$.

One property that is true of all[3] hyper-power functions is $x\,\boxed{n}\,2 = x\,\boxed{n+1}\,x$

2.2.4 Hyper-roots

A **hyper-root** is the inverse function of a hyper-power.

2.3 Other hyper-operations

All this time we have been talking about the standard *right-associative* hyper-operations. If we instead look at *left-associative* or even balanced iteration for the definition of operation succession, then we get different sets of hyper-operations.

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2.3.1 Munafo's lower hyper-operations

$$b_{(n)}c := (b_{(n)}(c-1))_{(n-1)}c \tag{2.17}$$

TODO

2.3.2 Bromer's mixed hyper-operations

TODO

2.3.3 Frappier's balanced hyper-operations

$$HF_n(b, c) := (x \mapsto HF_{n-1}(x, x))^{\circ \log_2(c)}(b) \tag{2.18}$$

TODO

2.3.4 Robbins' exponential hyper-operations

$$HR_n(b, c) = \exp_{\boldsymbol{e}} HR_{n-1}(\log_{\boldsymbol{e}}(b), \exp_{\boldsymbol{e}}^{\circ n-3}(c)) \tag{2.19}$$

TODO

2.3.5 Trappmann's commutative hyper-operations

$$HT_n(b, c) = \exp_{\boldsymbol{e}} HT_{n-1}(\log_{\boldsymbol{e}}(b), \log_{\boldsymbol{e}}(c)) \tag{2.20}$$

TODO

2.3.6 Trappmann's binary tree arithmetic

TODO

Chapter 3

Iteration theory

Here is an overview of some of the functions we will discuss.

$f(x)$	$f^{\circ t}(x)$	$\mathcal{A}[f]$ (Abel)	$\mathcal{S}[f]$ (Schröder)	$\mathcal{J}[f]$ (Julia)
$z+x$	$zt+x$	x/z	$a^{x/z}$	z
$z+ax$	$z\frac{a^t-1}{a-1}+a^t x$	$\log_a\left(\frac{z}{a-1}+x\right)$	$\frac{z}{a-1}+x$	$\left(\frac{z}{a-1}+x\right)\ln(a)$
ax	$a^t x$	$\log_a(x)$	x	$x\ln(a)$
x^y	x^{y^t}	$\log_y(\ln(x))$	$\ln(x)$	$x\ln(x)\ln(y)$
y^x	$\exp_y^{\circ t}(x)$	$\mathrm{slog}_y(x)$	$y^{\mathrm{slog}_y(x)}$	?
y^x-1	$\mathrm{dxp}_y^{\circ t}(x)$	?	?	$\ln(\ln(y))$?

3.1 Dynamical systems

3.1.1 Fixed points

Just as Taylor series of $f(x)$ can be described around any point with powers of the form $(x-x_0)^n$ we can just as well as describe the same series of $f(x+x_0)$ with powers of the form x^n without any loss of information. The same is true of fixed points. If $x_0 = f(x_0)$ is a fixed point of f, then we can always form a new function that behaves similar under iteration, such as $g(x) = f(x+x_0) - x_0$. Iterating this new function gives $g^{\circ n}(x) = f^{\circ n}(x+x_0) - x_0$, in which there is no loss of information.

There are two major conventions for denoting the coefficients of a power series. One is the subscript form, and the other is the alphabetic form:

$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots \tag{3.1}$$
$$= z + ax + bx^2 + cx^3 + dx^4 + \cdots \tag{3.2}$$

and one common thread to both conventions is that they both take the factorials into account. So the relationship between the subscript form and derivatives is $n!f_n = \left[\frac{\partial^n f}{\partial x^n}\right]_{x=0}$, and the relationship to convert this to the alphabetic form is to replace f_n with the n-th letter of the Roman alphabet.

3.1.2 Lyapunov number

3.2 Functional equations

3.2.1 Abel functions

The **principal Abel function** of f is defined as:

$$\mathcal{A}[f](x) := \lim_{n \to \infty} \left(\log_a f^{\circ n}(x) - n\right) \tag{3.3}$$

where the limit exists, and $a = f'(0)$ as indicated previously. Note that the limit cannot be distributed across the subtraction, since $\lim_{n \to \infty} n = \infty$ which does not converge, so the separate limits do not exist. In many cases, however, the combined limit does exist, and when it does it can be used to solve the functional equation:

$$\mathcal{A}[f](f(x)) = \mathcal{A}[f](x) + 1 \tag{3.4}$$

of which any solution can be called an Abel function of f. To see how this works for iterated multiples (exponentiation), if $f(x) = ax$, then $f^{\circ n}(x) = xa^n$, so:

$$\mathcal{A}[f](x) = \lim_{n \to \infty} \left(\log_a(xa^n) - n\right) \tag{3.5}$$

$$= \lim_{n \to \infty} \left(\log_a(x) + n - n\right) \tag{3.6}$$

$$= \lim_{n \to \infty} \left(\log_a(x)\right) \tag{3.7}$$

$$= \log_a(x) \tag{3.8}$$

which satisfies the functional equation: $\log_a(ax) = \log_a(x) + 1$.

The derivative of the Abel function of f can be expressed as:

$$\frac{\partial}{\partial x}\mathcal{A}[f](x) = \frac{1}{\mathcal{J}[f](x)} \tag{3.9}$$

so, in terms of the Julia function of f, the Abel function of f is:

$$\mathcal{A}[f](x) = \int_{x_0}^{x} \frac{dt}{\mathcal{J}[f](t)} \tag{3.10}$$

where x_0 is some chosen constant.

3.2.2 Schröder function

The **principal Schröder function** of f is defined as:

$$\mathcal{S}[f](x) := \lim_{n \to \infty} \left(\frac{f^{\circ n}(x)}{a^n} \right) \tag{3.11}$$

where the limit exists, and $a = f'(0)$. This is related to the Abel function of f by:

$$\mathcal{S}[f](x) := a^{\mathcal{A}[f](x)} \tag{3.12}$$

3.2.3 Böttcher function

$$\mathcal{B}[f](x) := a^{\mathcal{S}[f](x)} \tag{3.13}$$

3.2.4 Julia function

The **principal Julia function** is defined as:

$$\mathcal{J}[f](x) := \left[\frac{\partial}{\partial t} f^{\circ t}(x) \right]_{t=0} \tag{3.14}$$

which can be shown to be related the the Abel function of f by:

$$\mathcal{J}[f](x) = \frac{1}{\frac{\partial}{\partial x} \mathcal{A}[f](x)} \tag{3.15}$$

where the Abel function of f is non-constant.
  but there is more

$$\mathcal{J}[f](f^{\circ t}(x)) = f'(x)\mathcal{J}[f](x) \tag{3.16}$$

$$\mathcal{J}[f^{\circ t}](x) = t\mathcal{J}[f](x) \tag{3.17}$$

Open Question 3.18. *The Julia function of* $\mathrm{dxp}_b(x) = b^x - 1$ *is of the form*

$$\mathcal{J}_x[e^{ax} - 1](x) = \ln(a) \sum_{k=0}^{\infty} \frac{x^k \sum_{i=0}^{\binom{k-2}{2}} C_{ik} a^i}{\prod_{j=1}^{k-1}(a^j - 1)} \tag{3.19}$$

but so far there is no proof, and we do not know the C_{ik}. *What are the* C_{ik}?

3.3 Special matrices

The special matrices in this section are all related to power series expansions of functions, so we will start with how to analyze (break apart) a function into matrices, and to synthesize (put together) these matrices back into a function. Starting with the standard notation for power series (3.1) we can write a function as:

$$f(x) = \sum_{k=0}^{\infty} f_k x^k = \mathbf{F}[f]^T \mathbf{V}(x) = \begin{bmatrix} f_0 & f_1 & f_2 & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \end{bmatrix} \tag{3.20}$$

where $\mathbf{F}[f]^T$ is the coefficients of f, and $\mathbf{V}(x)$ is a Vandermonde vector of x.

There is an important distinction between two types of such powerseries, $f_0 = 0$ and $f_0 \neq 0$. These series behave in regard of iteration much differently. Iterations of f, where the value of the function is reintroduced as an argument, as in $f(x)$, $f(f(x))$, $f(f(f(x)))$, can be denoted by $f^{\circ n}(x)$. This discussion considers f as a formal power series, so considerations of convergence can be safely ignored.

To compute the iterates of $f(x)$, we can look first to its second iterate:

$$f^{\circ 2}(x) = \tag{3.21}$$

3.3.1 Carleman matrix

Using the vectors we defined above, we can define the Carleman matrix of f as:

$$\mathbf{M}[f] = \begin{bmatrix} \mathbf{F}[1]^T \\ \mathbf{F}[f]^T \\ \mathbf{F}[f^2]^T \\ \vdots \end{bmatrix} \tag{3.22}$$

in which each row is a vector of the coefficients of the j-th power of f (not iterate). We can also reference the elements individually to define it as:

$$\mathbf{M}[f]_{jk} = \frac{1}{k!} \left[\frac{\partial^k}{\partial x^k} (f(x))^j \right]_{x=0} \tag{3.23}$$

which satisfies the functional equation:

$$(f(x))^j = \sum_{k=0}^{\infty} \mathbf{M}[f]_{jk} x^k \tag{3.24}$$

17

Going back to the vector-based definition of the Carleman matrix, we can see that:

$$\mathbf{M}[f]\mathbf{V}(x) = \begin{bmatrix} \mathbf{F}[1] \cdot \mathbf{V}(x) \\ \mathbf{F}[f] \cdot \mathbf{V}(x) \\ \mathbf{F}[f^2] \cdot \mathbf{V}(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ f(x) \\ f(x)^2 \\ \vdots \end{bmatrix} = \mathbf{V}(f(x)) \tag{3.25}$$

which implies the fundamental property of Carleman matrices: converting function iteration into matrix multiplication. By replacing $f(x) \to f^{\circ n}(x)$, we get:

$$\mathbf{V}(f^{\circ n}(x)) = \mathbf{M}[f^{\circ n}]\mathbf{V}(x) \tag{3.26}$$

and by replacing $x \to f^{\circ n-1}(x)$ and reducing, we get:

$$\mathbf{V}(f^{\circ n}(x)) = \mathbf{M}[f]\mathbf{V}(f^{\circ n-1}(x)) \tag{3.27}$$
$$= \mathbf{M}[f]^2\mathbf{V}(f^{\circ n-2}(x)) \tag{3.28}$$
$$= \mathbf{M}[f]^n\mathbf{V}(x) \tag{3.29}$$

which proves the fundamental property:

$$\mathbf{M}[f^{\circ n}] = \mathbf{M}[f]^n \tag{3.30}$$

There are two cases when the Carleman matrix is triangular: linear functions:

$$\mathbf{M}_x[z + ax] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ z & a & 0 & 0 & 0 & \cdots \\ z^2 & 2za & a^2 & 0 & 0 & \cdots \\ z^3 & 3z^2a & 3za^2 & a^3 & 0 & \cdots \\ z^4 & 4z^3a & 6z^2a^2 & 4za^3 & a^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.31}$$

and functions such as $f(x) = \sum_{k=1}^{\infty} f_k x^k$ where $z = 0$ is a fixed point:

$$\mathbf{M}[f] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a & b & c & d & \cdots \\ 0 & 0 & a^2 & 2ab & 2ac + b^2 & \cdots \\ 0 & 0 & 0 & a^3 & 3a^2b & \cdots \\ 0 & 0 & 0 & 0 & a^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.32}$$

18

3.3.2 Bell matrix

$$\mathbf{B}[f] = \mathbf{M}[f]^T \tag{3.33}$$

So

$$\mathbf{B}[f] = \begin{bmatrix} \mathbf{F}[1] & \mathbf{F}[f] & \mathbf{F}[f^2] & \cdots \end{bmatrix} \tag{3.34}$$

We can also reference the elements individually to define it as:

$$\mathbf{B}[f]_{jk} = \frac{1}{j!} \left[\frac{\partial^j}{\partial x^j} (f(x))^k \right]_{x=0} \tag{3.35}$$

3.3.3 Abel matrix

Let

$$\mathbf{1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The **Abel matrix** of f (of size n) is defined as

$$\mathbf{A}[f] := \mathbf{J}(\mathbf{B}[f] - \mathbf{I})\mathbf{K} \tag{3.36}$$

This can be visualized as the upper-right submatrix of the Bell matrix of f, that is the matrix that includes all elements except the first column and the last row.

The power of the Abel matrix is that it encodes the Abel functional equation, or in other words, if $\mathbf{A}[f]\mathbf{F}[\alpha] = \mathbf{1}$, then $\mathcal{A}[f](x) = \alpha(x) = (\mathbf{F}[\alpha] \cdot \mathbf{V}(x))x + C$, where the extra x has been added because the coefficient α_1 is now the first element of $\mathbf{F}[\alpha]$ and C is any constant. To illustrate with $f(x) = z + ax + bx^2 + \cdots$:

$$\mathbf{A}[f]\mathbf{F}[\alpha] = \begin{bmatrix} z & z^2 & z^3 & \cdots \\ a-1 & 2az & 3az^2 & \cdots \\ b & a^2+2bz-1 & 3a^2z+3bz^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{bmatrix} \tag{3.37}$$

which implies that $\mathbf{F}[\alpha] = \mathbf{A}[f]^{-1}\mathbf{1}$ if the Abel matrix is invertible.

3.4 Regular iteration

3.4.1 General fixed points $(z = 0)$

According to Daniel Geisler [?]:

$$f^{\circ t}(x) = a^t x$$

$$+ \left(b \sum_{k_1=0}^{t-1} a^{2t-k_1-2} \right) \frac{x^2}{2!}$$

$$+ \left(c \sum_{k_1=0}^{t-1} a^{3t-2k_1-3} + 3b^2 \sum_{k_1=0}^{t-1} \sum_{k_2=0}^{t-k_1-2} a^{3t-2k_1-k_2-5} \right) \frac{x^3}{3!}$$

$$+ \cdots$$

3.4.2 Non-parabolic fixed points $(z = 0, a \neq 1)$

$$f^{\circ t}(x) = a^t x$$

$$+ \left(b \frac{a^{t-1}(a^t - 1)}{(a-1)} \right) x^2$$

$$+ \left(c \frac{a^{t-1}(a^{2t} - 1)}{(a^2 - 1)} + b^2 \frac{2a^{t-2}(a^t - a)(a^t - 1)}{(a-1)(a^2 - 1)} \right) x^3$$

$$+ \left(d \frac{a^{t-1}(a^{3t} - 1)}{(a^3 - 1)} \right.$$

$$+ bc \frac{a^{t-2}(a^t - a)(a^t - 1)(a^t(5 + 3a) + 5a + 2)}{(a^2 - 1)(a^3 - 1)}$$

$$\left. + b^3 \frac{a^{t-3}(a^t - a)(a^t - 1)(a^t(5 + a) - 5a^2 - 1)}{(a-1)(a^2 - 1)(a^3 - 1)} \right) x^4$$

$$+ \cdots$$

3.4.3 Parabolic fixed points $(z = 0, a = 1)$

$$
\begin{aligned}
f^{\circ t}(x) = \ & x \\
& + btx^2 \\
& + \left(ct + b^2 t(t-1)\right) x^3 \\
& + \left(dt + bc\frac{5}{2}t(t-1) + b^3\frac{1}{2}t(t-1)(2t-3)\right) x^4 \\
& + \left(et + (2bd + c^2)\frac{3}{2}t(t-1) + b^2 c\frac{1}{6}t(t-1)(26t-37)\right. \\
& \left. \quad + b^4\frac{1}{3}t(t-1)(t-2)(3t-4)\right) x^5 \\
& + \cdots
\end{aligned}
$$

3.5 Natural iteration

Natural iteration is based entirely on the Abel matrix of f. After solving the matrix equation $\mathbf{A}[f]\mathbf{F}[\alpha] = \mathbf{1}$, we obtain the Taylor coefficients of $\alpha(x)$, which can be solved for symbolically. Starting with $f(x) = z + ax + bx^2 + \cdots$, we find that the solutions for the first few approximants $(n = 1, 2, 3)$ are as follows:

$$
\alpha(x)_1 = \frac{x}{z} \tag{3.38}
$$

$$
\alpha(x)_2 = \frac{2azx + (1-a)x^2}{z^2(a+1)} \tag{3.39}
$$

$$
\alpha(x)_3 = \frac{3a(a^2+1)z^2x + 3(bz - a^3 + a^2)zx^2 + (-2bz + a^3 - a^2 - a + 1)x^3}{z^3(bz + a^3 + 2a^2 + 2a + 1)} \tag{3.40}
$$

21

Chapter 4

Tetration

Question 4.1. *How is tetration written?*

Tetration can be represented in various other notations as:

$$
\begin{aligned}
x^{x^{\cdot^{\cdot^{\cdot^{x^x}}}}} &= {}^n x && \text{(Maurer's left-superscript notation)} \\
&= x\,\boxed{4}\,n && \text{(Romerio's box notation)} \\
&= x\!\uparrow\!\uparrow\! n && \text{(Knuth's up arrow notation)} \\
&= (x \to n \to 2) && \text{(Conway's chained arrow notation)} \\
&= \exp_x^n(1) && \text{(iterated exponential notation)} \\
&= \prod_{k=1}^{n} x = \mathrm{T}^n x && \text{(nested exponential notation)}
\end{aligned}
$$

4.1   History of tetration

The history of tetration starts with Euler, Eissenstein, Goodstein, Maurer, Lambert, and many other people, depending on which aspect of tetration one is considering. Also, various functions related to tetration, such as iterated exponentials and nested exponentials have been developed by different people at different times.

4.1.1   Iterated exponentials

Question 4.2. *How are iterated exponentials written?*

Iterated exponentials can be represented in various notations as:

$$b^{b^{\cdot^{\cdot^{b^x}}}} \begin{aligned} &= \exp_b^n(x) && \text{(Euler's mnemonic notation)} \\ &= (b\uparrow)^n(x) && \text{(Knuth's notation)} \\ &= {}^n(b,x) && \text{(Galidakis' notation)} \\ &= {}^n b^x && \text{(Robbins' notation)} \\ &= b^{[n]}(x) && \text{(Campagnolo \textit{et.al.} notation)} \\ &= \underset{k=1}{\overset{n}{\mathrm{T}}}(b;x) = \mathrm{T}^n(b;x) && \text{(nested exponential notation)} \end{aligned}$$

4.1.2 Nested exponentials

Question 4.3. *How are nested exponentials written?*

Nested exponentials can be represented as:

$$a_1^{a_2^{\cdot^{\cdot^{a_{n-1}^{a_n}}}}} \begin{aligned} &= \mathrm{T}(a_1, a_2, \cdots, a_{n-1}, a_n) && \text{(Barrow-Brunson notation)} \\ &= \underset{k=1}{\overset{n-1}{\mathrm{T}}}(a_k; a_n) && \text{(Barrow-Shell notation)} \\ &= \underset{k=1}{\overset{n}{\mathrm{T}}} a_k && \text{(Barrow's notation)} \end{aligned}$$

Historical notations and terminology

- D. F. Barrow uses the notation $E(a_1, a_2, a_3, \cdots, a_n) = \mathrm{T}_{k=1}^n a_k$ and the notation $E_{k=1}^\infty a_k = \mathrm{T}_{k=1}^\infty a_k$ where he coins the term *infinite exponentials* along with many others including the *n*th *margerin function* for $z \mapsto \mathrm{T}_{k=1}^n(a_k; z)$ in [15].

- D. L. Shell uses most of Barrow's terms, the notation $E(a; z) = \exp_a^\infty(z)$ and the notation $E(a_1, a_2, a_3, \cdots; z) = \mathrm{T}_{k=1}^\infty(a_k; z)$ in [**?**]

- W. J. Thron uses the notation $T_n(z) = \mathrm{T}_{k=1}^n(e^{a_k}; z)$ in [92].

- G. Bachman uses the notations $E_n = [a_1, a_2, a_3, \cdots, a_n] = \mathrm{T}_{k=1}^n a_k$ in [**?**].

- X. Dong uses the notation $E_{\lambda_1, \cdots, \lambda_n}(z) = \mathrm{T}_{k=1}^n(e^{\lambda_k}; z)$ in [**?**].

- B. W. Brunson uses the notation $T(a_1, a_2, a_3, \cdots, a_n) = \mathrm{T}_{k=1}^n a_k$ in [**?**].

- V. I. Yukalov and S. Gluzman use $F_n(t, \tau) = a_0 \mathrm{T}_{k=1}^n(e^{a_k/a_{k-1}}; \tau)$ in [105].

23

- S. Sykora uses the notations $E_{\pm}^n(x) = \{b_1 b_2 b_3 \cdots \mid x\} = \mathrm{T}_{k=0}^{\infty}(\beta^{2b_k - 1}; x)$ where $b_k \in \{0, 1\}$ and uses the term *binary iterated powers* (misnomer) in [90]. Its acronym **bips** sounds nice, though, so it might be used instead.

- N. J. A. Sloane uses the term *tower of height n* for $\mathrm{T}_{k=1}^n a_k$ in [88].

Using Barrow's terms, Shell, Thron, Dong, Sykora, and Yukalov *et.al.* are actually discussing *margerin exponential* functions, although they do not say so explicitly. This provides counter examples for the argument that nested exponentials are too rare to deserve a notation. Clearly, this list would indicate otherwise.

Although E is more commonly used than T, it is easily mistakable for Σ, especially when they appear together (for example: E Σ T Π), where it can be seen that it is much easier to mistake E for Σ than it is to mistake T for Π. It is for this reason that we use T in this Reference instead of E.

4.1.3 Exponential forests

There is a problem posted on a wall of the math club at the University of Maryland, by the author Andrew Snowden. He apparently teaches both there and at Princeton, and poses many interesting problems in mathematics. The problem set was simply called *Some More Problems*, and contained problems from algebra and number theory to real and complex analysis. The second problem posted seemed interesting, as it related to nested exponentials, so it will be reproduced here:

> A natural nummber n may be factored as $n = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ where the p_n are distinct prime numbers and a_n are natural numbers. Since the a_n are natural numbers, they may be factored in such a manner as well. This process may be continued, building a "factorization tree" until all the top numbers are 1. Thus any question that can be asked of trees (i.e. the height of a tree, the number of nodes in a tree, etc.) may be asked of our natural number n. This problem is about the height of n which we denote $h(n)$. Define:
>
> $$D_n = \lim_{N \to \infty} \frac{1}{N} |\{k \leq N : h(k) \geq n\}|$$
>
> D_n is sort of the density of numbers with height at least n. It is obvious that $D_1 = 1$ since all numbers have height at least 1.
>
> 1. Show that
>
> $$\frac{1}{2} < (^n 2) D_n \leq 3$$

24

2. Let a be the average height of a natural number (i.e. if you were to pick many numbers at random their height would average out to a). Using the previous part and other methods, give bounds on a. The best bounds [Andrew Snowden has found] are

$$1.42333 < a < 1.4618 \tag{4.4}$$

Here are some examples of *prime factorization trees* as suggested by Snowden:

Number	Factorization
4	2^2
8	2^3
9	3^2
12	$2^2 \cdot 3$
16	2^{2^2}
48	$2^{2^2} \cdot 3$
64	$2^{2 \cdot 3}$
81	3^{2^2}
144	$2^{2^2} \cdot 3^2$
4096	$2^{2^2 \cdot 3}$
65536	$2^{2^{2^2}}$
531441	$3^{2^2 \cdot 3}$
7625597484987	3^{3^3}

4.2 Standard definitions

Most of the standard definitions of tetration define it recursively:

$$^{1}x := x \tag{4.5}$$
$$^{n+1}x := x^{(^{n}x)} \tag{4.6}$$

such that tetration is only defined for natural number n. There are extensions of tetration beyond this domain, but each has a specific domain of applicability. For all complex number x within the Shell-Thron region, one can use regular iteration, and for all complex number x with real part greater than 1, one can use natural iteration, but for other bases, no extension has been found.

4.2.1 Super-exponentials (tetrationals)

Definition 4.7. The base-b **tetrational** function is $x \mapsto {}^{x}b$.

For some bases, no fancy methods are required, for example, base 0 and base 1. For base-0 tetrational functions,

4.2.2 Super-logarithms

Definition 4.8. The base-b **super-logarithm** is the inverse of the base-b tetrational.

Using the third interpretation of super-logarithms (i.e. the number of times you must apply a logarithm before one is obtained), we can define integer versions as:

$$\lfloor \mathrm{slog}_x(z) \rfloor = \begin{cases} -2 & \text{if } z < 0, \\ \lfloor \mathrm{slog}_x(\log_x(z)) \rfloor + 1 & \text{if } z \geq 0. \end{cases} \tag{4.9}$$

$$\lceil \mathrm{slog}_x(z) \rceil = \begin{cases} -1 & \text{if } z \leq 0, \\ \lceil \mathrm{slog}_x(\log_x(z)) \rceil + 1 & \text{if } z > 0. \end{cases} \tag{4.10}$$

[superlogarithm base 1] [superlogarithm base infty]

4.2.3 Super-powers (tetrates)

Definition 4.11. The n-th **tetrate** function is $x \mapsto {}^{n}x$.

Theorem 4.12 (Galidakis). *For $x \in \mathbb{C}$ and $0 \leq n \in \mathbb{Z}$:*

$$ {}^{n}x = \sum_{k=0}^{\infty} p_{nk} \log(x)^k \tag{4.13}$$

where:

$$p_{nk} = \begin{cases} 1 & \text{if } n \geq 0 \text{ and } k = 0, \\ 0 & \text{if } n = 0 \text{ and } k > 0, \\ \frac{1}{k!} & \text{if } n = 1, \end{cases} \tag{4.14}$$

otherwise:

$$\boxed{p_{nk} = \frac{1}{k} \sum_{j=1}^{k} j p_{n(k-j)} p_{(n-1)(j-1)}} \tag{4.15}$$

and Galidakis gives the recurrence equation (4.15) in [?].

Theorem 4.16 (Robbins)**.** *For $x \in \mathbb{C}$ and $0 \leq n \in \mathbb{Z}$:*

$$^{n}x = \sum_{k=0}^{\infty} t_{nk}(x-1)^{k} \tag{4.17}$$

where:

$$t_{nk} = \begin{cases} 1 & \text{if } n \geq 0 \text{ and } k = 0, \\ 0 & \text{if } n = 0 \text{ and } k > 0, \\ 1 & \text{if } n = 1 \text{ and } k = 1, \\ 0 & \text{if } n = 1 \text{ and } k > 1, \end{cases} \tag{4.18}$$

otherwise:

$$\boxed{t_{nk} = \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=j}^{k} i(-1)^{j-1} t_{n(k-i)} t_{(n-1)(i-j)}} \tag{4.19}$$

4.2.4 Super-roots

Definition 4.20. The n-th **super-root** is $\text{srt}_n(z) = x$ if and only if $^{n}x = z$.

A *super-root* is an inverse function of tetration, or to be more specific, the inverse of a hyperpower function.

Before we can go into more detail about series expansions of super-roots, we must first recall the process of finding a power series for an inverse function, given a power series for the function itself. This process is called Lagrange inversion [?] (sometimes confused with Lagrange reversion). Lagrange inverse series:

4.2.5 Iterated exponentials

Definition 4.21. The n-th base-b **iterated exponential** is $\exp_b^n(x)$.

Generally speaking, there are three variables in a iterated exponential, so there are at least three different ways of forming a power series expansion of them. For series expansions of $\exp_b^c(a)$, series about $a = (^{\infty}b)$ are usually discussed in *regular iteration*, and series about $c = 0$ are usually discussed in *natural iteration* but series in b are specific to tetration, so these expansions are covered here.

Theorem 4.22 (Galidakis)**.** *For $x \in \mathbb{C}$, $z \in \mathbb{C}$ and $0 \leq n \in \mathbb{Z}$:*

$$^{n}(x; z) = \sum_{k=0}^{\infty} p_{nk}(z) \log(x)^{k} \tag{4.23}$$

27

where:

$$p_{nk}(z) = \begin{cases} 1 & \text{if } n > 0 \text{ and } k = 0, \\ z & \text{if } n = 0 \text{ and } k = 0, \\ 0 & \text{if } n = 0 \text{ and } k > 0, \\ \frac{z^k}{k!} & \text{if } n = 1, \end{cases} \tag{4.24}$$

otherwise:

$$\boxed{p_{nk}(z) = \frac{1}{k} \sum_{j=1}^{k} j p_{n(k-j)}(z) p_{(n-1)(j-1)}(z)} \tag{4.25}$$

Theorem 4.26 (Robbins)**.** *For $x \in \mathbb{C}$, $z \in \mathbb{C}$ and $0 \le n \in \mathbb{Z}$:*

$$^n(x; z) = \sum_{k=0}^{\infty} t_{nk}(z)(x-1)^k \tag{4.27}$$

where:

$$t_{nk}(z) = \begin{cases} 1 & \text{if } n > 0 \text{ and } k = 0, \\ z & \text{if } n = 0 \text{ and } k = 0, \\ 0 & \text{if } n = 0 \text{ and } k > 0, \\ \binom{z}{k} & \text{if } n = 1, \end{cases} \tag{4.28}$$

otherwise:

$$\boxed{t_{nk}(z) = \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=j}^{k} i(-1)^{j-1} t_{n(k-i)}(z) t_{(n-1)(i-j)}(z)} \tag{4.29}$$

4.2.6   Auxiliary super-roots

Definition 4.30. The n-th **super-root from** x is $\mathrm{srt}_n(z; x) = b$ where $z = \exp_b^n(x)$.

For more information about auxiliary super-roots, see appendix C.

4.3   Standard extensions

The reason why two extensions of tetration are singled out is that *neither* of these are extensions of tetration but *methods of iteration*. Since both *regular iteration* and *natural iteration* apply to almost any function, they are very powerful tools outside of tetration, and they also apply to tetration quite well. To compare these methods, regular tetration is a power series expansion in the *initial/top/auxiliary* exponent, while natural tetration is a power series expansion in the *height*.

4.3.1 Embedded tetration (Aldrovandi-Helms)

TODO

4.3.2 Regular tetration (Schröder-Szekeres-Geisler)

TODO

4.3.3 Natural tetration (Abel-Walker-Robbins-Fox)

4.4 Other methods

4.4.1 Approximations

TODO

4.4.2 Ingolf Dahl's method (fractional)

TODO

4.4.3 Jay D. Fox's method (linear)

TODO

4.4.4 Ioannis Galidakis' method (bump)

The second extension Galidakis gives in [?] is defined as follows.

Definition 4.31 (Galidakis). If $x \in \mathbb{C}$ and $y \in \mathbb{R}$:

$$^{y}x \approx \mathrm{IG}(x,y) = \sum_{k=0}^{\infty} \alpha_k(y) \log(x)^k \tag{4.32}$$

where:

$$\alpha_k(y) = \begin{cases} 1 & \text{if } k = 0, \\ \int_0^y \sum_{j=1}^k \frac{(P_{jk} - P_{(j-1)k})f(y-j+\frac{1}{2})}{\int_{n-1}^n f(s-j+\frac{1}{2})\boldsymbol{ds}}\boldsymbol{dt} & \text{otherwise.} \end{cases} \tag{4.33}$$

and:

$$f(x) = \begin{cases} \exp\left(\frac{4}{4x^2-1}\right) & \text{if } |x| < 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.34}$$

and:
$$P_{nk} = \begin{cases} 1 & \text{if } n \geq 0 \text{ and } k = 0, \\ 0 & \text{if } n = 0 \text{ and } k > 0, \\ \frac{1}{k!} & \text{if } n = 1, \end{cases} \tag{4.35}$$

otherwise:
$$P_{nk} = \frac{1}{k} \sum_{j=1}^{n} j P_{n(k-j)} P_{(n-1)(j-1)} \tag{4.36}$$

One advantage of (IG) is that it utilizes (4.36), a remarkable recurrence equation, but uses a bump function in (4.34) that seems unnatural.

4.4.5 Robert Munafo's method (tower)

Munafo's extension of tetration is a nested exponential, which can be expressed in terms of the n-ary exponential notation introduced in an earlier chapter.

$$g(x, z) = x^{\frac{1}{2}(1 + \text{erf}(2z - 1))} \tag{4.37}$$

$$g_0(x, y, n) = \prod_{k=(-n)}^{0} g(x, y + k) \tag{4.38}$$

$$g_1(x, y) = \lim_{n \to \infty} g_0(x, y, n) \tag{4.39}$$

$$g_2(x, y, m) = \log_x^m(g_1(x, y + m)) \tag{4.40}$$

$$g_3(x, y) = \lim_{m \to \infty} g_2(x, y, m) \tag{4.41}$$

$$RM(x, y) = g_3(x, y + K(x)) \tag{4.42}$$

$$^y x \approx RM(x, y) \tag{4.43}$$

where $K(x)$ is a number such that $RM(x, 1) = x$. Summarizing:

$$^y x \approx RM(x, y) = \lim_{m \to \infty} \log_x^m \left(\lim_{n \to \infty} \prod_{k=(-n)}^{0} x^{\frac{1}{2}(1 + \text{erf}(2(y + k + m + K(x)) - 1))} \right) \tag{4.44}$$

Nelson's and Munafo's extensions are unique in that they do not immediately imply a series expansion, but Galidakis' and Geisler's extensions are series expansions, so we can (in theory) compare them, along with other approximations.

4.4.6 Clifford Nelson's method (recursive)

In a conversation posted in the Math Atlas [81], Clifford Nelson and Seth Russell discuss the extension of hyper-operations using a recursive definition that allows non-integer hyper-exponents to be calculated through hyper-logarithms. In this conversation, Nelson defines a transform of a function $f(x, y)$ as follows:

$$CN[f](x, y)_n = \begin{cases} 1 & \text{if } x = 0, \\ 1 & \text{if } y = 0, \\ 1 & \text{if } x = y, \\ x & \text{if } y = 1, \\ c & \text{if } n = 0 \text{ for any } c, \\ 1 + CN[f](f(x, y), y)_{n-1} & \text{if } x \geq y, \\ 1/CN[f](y, x)_{n-1} & \text{if } x < y \end{cases} \quad (4.45)$$

He then shows the following (using $c = 1$ for example):

$$CN_{xy}[x - y](x, y)_\infty = x/y \quad (4.46)$$
$$CN_{xy}[x/y](x, y)_\infty = \log_y(x) \quad (4.47)$$
$$CN_{xy}[\log_y(x)](x, y)_\infty \approx \mathrm{slog}_y(x) \quad (4.48)$$
$$CN_{xy}[\mathrm{slog}_y(x)](x, y)_\infty \approx {}_y^5\underline{x} \quad (4.49)$$

where the convention $CN_{xy}[f(x, y)] = CN[f]$ is used. So, Nelson's transform uses a hyper-n-logarithm $f(x, y)$ to find the hyper-$(n + 1)$-logarithm. The problem with this method, however, is that the super-logarithm obtained is not continuous.

4.4.7 Andrew Robbins's method (quadratic)

TODO

4.4.8 Peter Walker's method (iterative)

TODO

4.4.9 S. C. Woon's method (binomial)

TODO

Chapter 5

Special Functions

Most texts about special functions usually consider: Airy, Bessel, Chebyshev, and Lagendre functions, but here we will consider a different set of special funcitons. Instead of considering special functions that are the solutions of differential equations (as most are), we will be considering functions such as x^x.

Unlike the functions that are represented by single letters (like f or W) or perhaps 3 letter lowercase mnemonics (like exp, log, sin, cos, ...), this Reference uses a wide variety of two letter uppercase mnemonics. Here is an overview of some of the symbols used in this Reference, for more, see appendix A. The *new* terminologies introduced in this Reference are indicated in **bold**, even if there is one another place it has been used before.

Sym.	Function Name	Forms	Inverse	Min	Max
PE_e	base-e product-exponential	$x\boldsymbol{e}^x$	$W(x)$	-1	∞
W	Lambert W function	$W(x)$	$x\boldsymbol{e}^x$	$-1/\boldsymbol{e}$	∞
SP	**self-power**	$x^x = {}^2x$	${}^{2}_{4}\!\sqrt{x}$	$1/\boldsymbol{e}$	∞
SR	**self-root**	$x^{1/x} = \sqrt[x]{x} = {}^{\infty}_{4}\!\sqrt{x}$	${}^\infty x$	0	$\boldsymbol{e}$
	reciprocal self-power	$x^{-x} = \frac{1}{x^x} = (1/x)^x$	$1/{}^\infty x$	∞	$1/\boldsymbol{e}$
	reciprocal self-root	$x^{-1/x} = (1/x)^{(1/x)}$	${}^\infty(1/x)$	$\boldsymbol{e}$	0
ssqrt	super-sqrt	${}^{2}_{4}\!\sqrt{x} = 1/{}^\infty(1/x)$	x^x	$\boldsymbol{e}^{-1/\boldsymbol{e}}$	∞
H	**Knoebel H function**	${}^\infty x = 1/{}^{2}_{4}\!\sqrt{1/x}$	$x^{1/x}$	0	$\boldsymbol{e}^{1/\boldsymbol{e}}$
EC	**exponential commutator**	$EC(x) = {}^\infty(x^{1/x})$	$EC(x)$	∞	1
EF	exponential factorial	$EF(x) = x^{EF(x-1)}$		-1	∞

This chapter is supposed to be the most complete reference about these functions, if the information is scarce. For example, the gamma function is ubiquitous, so its discussion will be brief, but the super-sqrt is more rare (being found mostly in the work of Rubstov and Romerio) so for this, we provide a more complete treatise.

5.1 Named functions

5.1.1 Euler gamma function (factorial)

Definition 5.1. The **Euler Γ function** is

$$\Gamma(z+1) := \int_0^\infty t^z \boldsymbol{e}^{-t} \boldsymbol{dt}$$
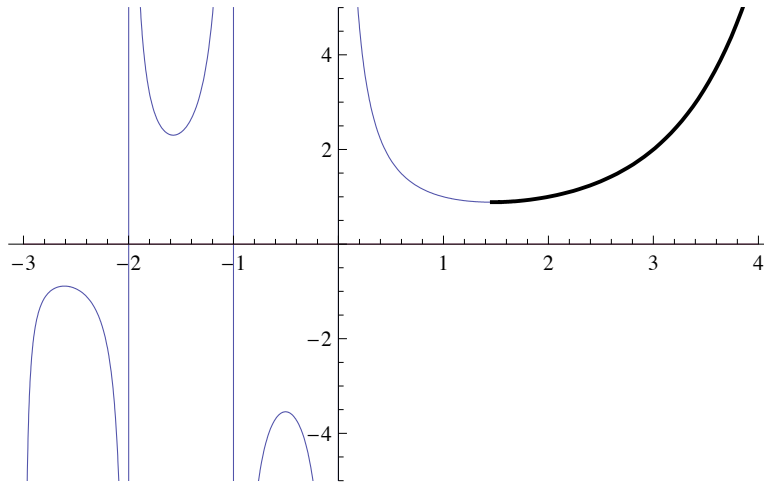
Figure 5.1: Plot of $\Gamma(x)$ (**invertible**, real, imaginary)

The gamma function is the unique solution to:

$$\left\{ \begin{array}{ll} \Gamma(1) = 1 & \\ \Gamma(z+1) = z\Gamma(z) & \forall z \in \mathbb{C} \\ \frac{d^2}{dz^2} \log \Gamma(z) \geq 0 & \forall z > 0 \end{array} \right\} \tag{5.2}$$

The gamma function is very well known, but mostly as the analytic continuation of the factorial, meaning it extends the following product to complex n:

$$n! = \prod_{k=1}^n k = \Gamma(n+1) \tag{5.3}$$

33

5.1.2 Barnes G function (super-factorial)

Definition 5.4. The **Barnes G function** is

$$G(z+1) := \sqrt{(2\pi)^z / \exp\left(z + z^2(1+\gamma)\right)} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right)^n \exp\left(\frac{z^2}{2n} - z\right) \right]$$
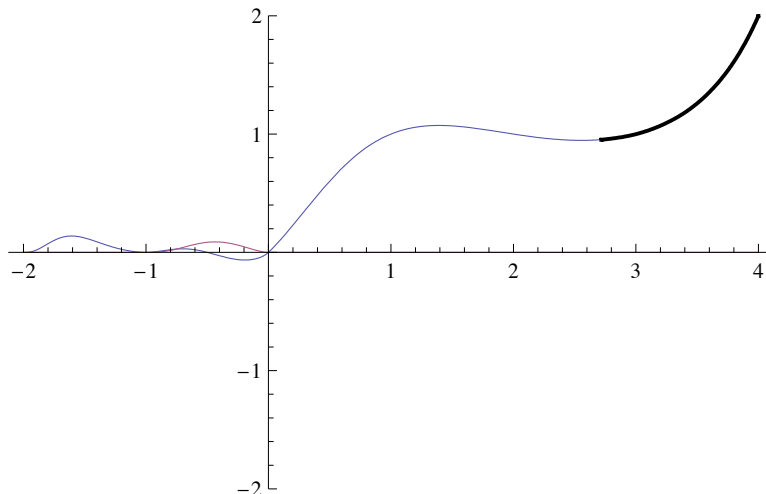
Figure 5.2: Plot of $G(x)$ (**invertible**, real, imaginary)

Another definition is $G(z+1) = \exp\left(z \log \Gamma(z) + \zeta'(-1) - \zeta'(-1, z)\right)$ where $\zeta(t, z)$ is the Hurwitz zeta function, and $\zeta'(t, z) = \frac{d}{dt}(t, z)$.

The Barnes G function is the unique solution to:

$$\left\{ \begin{array}{ll} G(1) = 1 & \\ G(z+1) = \Gamma(z)G(z) & \forall z \in \mathbb{C} \\ \frac{d^3}{dz^3} \log G(z) \geq 0 & \forall z > 0 \end{array} \right\} \tag{5.5}$$

The Barnes G function extends the following product to complex n:

$$n\$ = \prod_{k=1}^{n} k! = G(n+2) \tag{5.6}$$

where $n\$$ is the super-factorial.

References for the Barnes G function

1. V. S. Adamchik, *Contributions to the Theory of the Barnes Function*, <http://www-2.cs.cmu.edu/~adamchik/articles/barnes1.pdf>.

2. V. S. Adamchik, *Symbolic and Numeric Computation of the Barnes Function*, <http://math.unm.edu/ACA/2001/Proceedings/SymNum/Adamchik_paper.pdf>.

3. E. W. Barnes, *The Theory of the Double Gamma Function*, Philosophical Transactions of the Royal Society of London. ser. A, **196** (1901): 265-387.

4. E. W. Barnes, *The Theory of the G-function*, Quarterly Journal of Pure and Applied Mathematics **31** (1900): 265-314.

5. Ashkan Nikeghbali; Marc Yor, *The Barnes G function and its relations with sums and products of generalized Gamma convolution variables*, arXiv, <http://arxiv.org/abs/0707.3187v1>.

6. Eric W. Weisstein, *Barnes G-function*, MathWorld, <http://mathworld.wolfram.com/BarnesG-Function.html>.

7. *Barnes G-function*, <http://en.wikipedia.org/wiki/Barnes_G-function>.

5.1.3 Fox μ function (change of base formula)

TODO

5.1.4 Galidakis HW function (hyper-product-logarithm)

Definition 5.7. The **Galidakis HW function** is

$$HW_k(c_1, \cdots, c_n; \ z) = w \text{ where } w \prod_{k=1}^{n} (e^{c_k}; w) = z \tag{5.8}$$

According to Galidakis, a Taylor series expansion of an HW function is:

$$HW(\mathbf{c}; w) = z_0 + \sum_{n=1}^{\infty} \frac{(w - w_0)^n}{n!} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left[\frac{z - z_0}{G(z) - w_0} \right]^n \right\}_{z=z_0}$$

where

$$G(z) = z \prod_{n=1}^{m} (e^{c_n}; z)$$

which he derives using the Lagrange Inversion Theorem.

35

Figure 5.3: Plot of $H(x) = {}^\infty x$ (**converge**, real, imaginary)

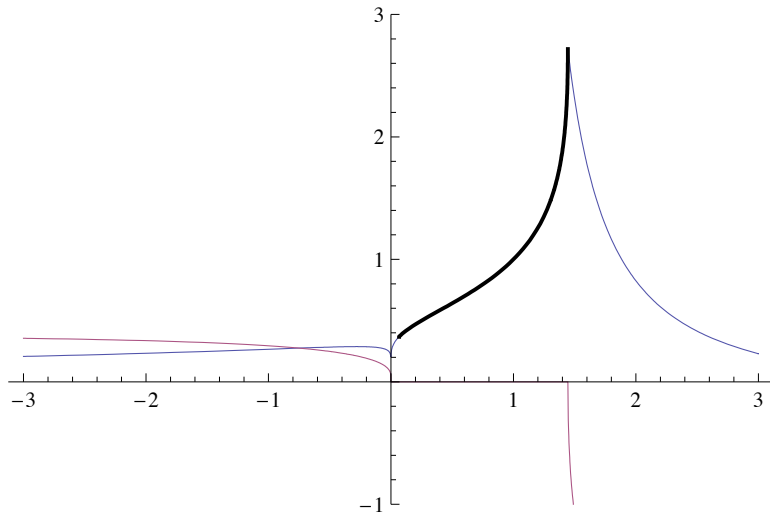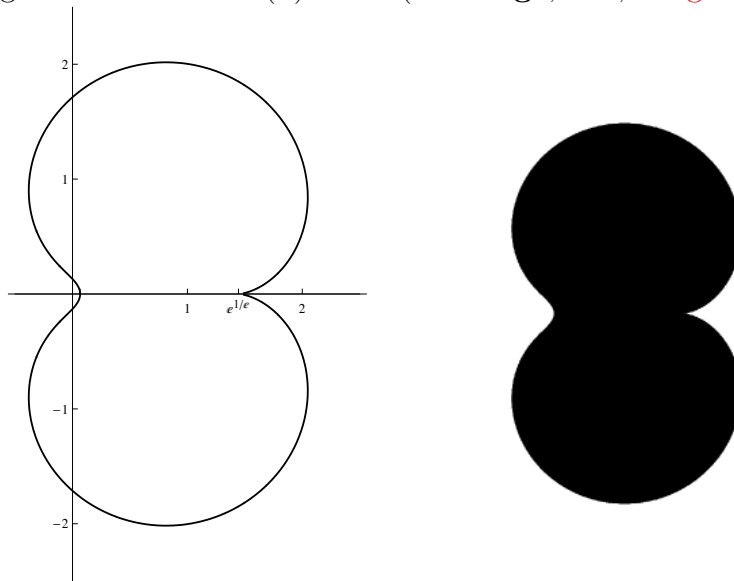Figure 5.4: Region of convergence of $\lim_{n \to \infty} {}^n x$

5.1.5 Knoebel H function (infinitely iterated exponential)

Definition 5.9. The **infinitely iterated exponential** is $H_L(x) = \lim\limits_{n \to \infty} {}^{n}x$.

Definition 5.10. The **Knoebel H function** is $H_k(x) = \exp\left(-W_{(-k)}(-\log(x))\right)$

Another version of this function by Galidakis is $h_k(x) = \exp\left(-W_k(-\log(x))\right)$ but this gives the wrong branch structure, because $h_0(x) < h_{(-1)}(x)$ whereas with the definition above $H_0(x) < H_1(x)$ which coincides with the real branch structure. We can use H and h to distinguish between the two branch structures of this function.

In Knoebel's 1981 paper [68], he asked three questions: *"When does $x^y < y^x$"*, *"When does $x^y = y^x$"* and *"Is there a formula for $y = f(x)$"*. For the first question, see the section on iterated exponentials. For the second and third question, see the section on the exponential commutator. The exponential commutator can be found through the different branches of the Knoebel H function, described here.

The infinitely iterated exponential (H_L) is the first form of tetration, and its convergence is the first theorem of tetration. In 1783, Euler proved that H_L converges on the positive real axis if and only if $e^{-e} \leq x \leq e^{1/e}$, later discussed by Eisenstein in 1844. Since both Euler and Eisenstein were the first to talk about H_L, this interval is sometimes known as the **Euler-Eisenstein interval**.

The infinitely iterated exponential spawned research into nested exponentials, where Barrow [15] and others sought converges tests comparable to those available for infinite series. Although few convergence tests exist, Bachman [?] concludes that with the addition of his proof of the **Ramanujan convergence test**, the theory of the convergence of infinitely nested exponentials is complete.

The theory of infinitely iterated exponentials had been worked on by Shell in his thesis [?]. Thron [?] had come close to a complete parameterization of the region in the complex plane where H_L converges, but left some points out. Shell proved that $H_L(e^{\zeta e^{-\zeta}})$ converges to e^{ζ} if $|\zeta| < 1$. This region is illustrated in figure (5.4), and is sometimes called the **Shell-Thron region**. Shell then went on to derive a convergence test for infinitely iterated exponentials with arbitrary top exponent, where he shows that $\exp^{\infty}_{(e^{\zeta e^{-\zeta}})}(z)$ converges to e^{ζ} if and only if $|\zeta| < 1$ and:

$$\boxed{|z - e^{\zeta}| < r|e^{\zeta}|} \tag{5.11}$$

where r is a positive root of the equation $\log(1 + r)/r = |\zeta|$.

A general derivative of the Knoebel H function is:

$$H'(x) = \frac{H(x)^2}{x(1 - H(x)\log(x))} \tag{5.12}$$

37

which can be turned into a differential equation by cross multiplying, which in turn can be solved by the Frobenius method to obtain the coefficients. Fortunately, the coefficients have already been found, and there are two well-known expansions for this function, one due to Eissenstein [**?**], and the other due to Jovovic.

A Puiseux series expansion of H is:

Theorem 5.13.

$$\infty x = \sum_{k=0}^{\infty} \frac{\log(x)^k}{k!}(k+1)^{(k-1)} \tag{5.14}$$

A Taylor series expansion of H is:

Theorem 5.15 (Jovovic [A033917])**.**

$$\infty x = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (j+1)^{(j-1)} \tag{5.16}$$

5.1.6 Lambert W function (product-logarithm)

Figure 5.5: Plot of $W(x)$ (**invertible**, real, imaginary)

38

Definition 5.17. The **Lambert W function** is defined by $W(x)e^{W(x)} = x$.

The Lambert W function is the most important function in its class, because it is available in most major computer algebra systems (CAS) as a built-in function. It is available in Maple as `LambertW(x)` and in Mathematica as `LambertW[x]`.

A closely related function is defined as $T(x) = -W(-x)$, which is known as the tree function.

A general derivative of the Lambert W function is:

$$W'(x) = \frac{W(x)}{x(1 + W(x))} \tag{5.18}$$

5.1.7   Munafo class function

In Munafo's website about *Large Numbers* [80] he describes a number class system. His description is based on the *perception class* of a real number, possibly large. The **Munafo class** system starts by defining class-0 numbers as 1 through 6. Munafo then gives an upper-bound for class-1 at about 10^6, and for class-2 numbers at about 10^{10^6}. This is obviously a super-logarithmic scale, so we can use the super-logarithm to simplify his description.

Let the Munafo class of n be denoted $MC(n)$. The class function satisfies:

$$\exp_{10}^{MC(n)-1}(6) < n \le \exp_{10}^{MC(n)}(6) \tag{5.19}$$

Replace 6 with $\exp_{10}^{\text{slog}_{10}(6)}(1)$, and add hyper-exponents:

$$\exp_{10}^{MC(n)-1}\left(\exp_{10}^{\text{slog}_{10}(6)}(1)\right) < n \le \exp_{10}^{MC(n)}\left(\exp_{10}^{\text{slog}_{10}(6)}(1)\right) \tag{5.20}$$

$$\exp_{10}^{MC(n)-1+\text{slog}_{10}(6)}(1) < n \le \exp_{10}^{MC(n)+\text{slog}_{10}(6)}(1) \tag{5.21}$$

Take the super-log of each side, then subtract $MC(n) + \text{slog}_{10}(6)$ from each side:

$$MC(n) - 1 + \text{slog}_{10}(6) < \text{slog}_{10}(n) \le MC(n) + \text{slog}_{10}(6) \tag{5.22}$$

$$-1 < \text{slog}_{10}(n) - \text{slog}_{10}(6) - MC(n) \le 0 \tag{5.23}$$

$$\lceil \text{slog}_{10}(n) - \text{slog}_{10}(6) - MC(n) \rceil = 0 \tag{5.24}$$

Since the middle is always between -1 and 0 (but never -1), the ceiling of the middle is always zero. Using (7.6), we can take $MC(n)$ outside the ceiling, because it is an integer, so we can summarize the Munafo class $MC(n)$ of a number n as:

$$MC(n) = \lceil \text{slog}_{10}(n) - \text{slog}_{10}(6) \rceil \tag{5.25}$$

Munafo class of named large numbers

$MC(n)$	$\text{slog}_{10}(n)$	n	Name
-1	-1	0	zero
0	0	1	one
0	0.39312	2	two
0	0.58836	3	three
0	0.70801	4	four
0	0.79068	5	five
0	0.85223	6	six
1	0.90044	7	seven
1	0.93957	8	eight
1	0.97220	9	nine
1	1	10	ten
1	1.39312	100	hundred
1	1.58836	1000	thousand
1	1.85223	10^6	million
2	1.97220	10^9	billion
2	2.01411	$^\pi e \approx 3.7149 \times 10^{10}$	e-tetra-pi
2	2.31648	$|M| \approx 8.0801 \times 10^{53}$	monster group order
2	2.36588	$136 \times 2^{256} \approx 1.5748 \times 10^{79}$	Eddington's number
2	2.39312	10^{100}	googol
3	3.39312	$10^{10^{100}}$	googol-plex

5.2 Special cases of exponentials

5.2.1 Decremented exponentials

5.2.2 Product exponentials

Product exponentials $PE_b(z) = zb^z$ are not very common, except in certain partial differential equations. When they arise, they usually arise with base e, so the form $PE(z) = PE_e(z)$ is the most common.

5.2.3 Scaled exponentials

Work on the dynamics of the "exponential map" rarely talk about $\exp_x(z) = x^z$. Daniel Geisler [?] has done some initial research into the dynamics of $\exp_x$. However, Devaney [?] and others have written extensive investigations into the map $SE_\lambda(z) =$

Figure 5.6: Plot of $x\boldsymbol{e}^x$ (**invertible**, real, imaginary)

$\lambda \boldsymbol{e}^z$. The dynamics of SE_λ do not immediately imply the same properties of $\exp_x$, but we can relate them by noticing that:

$$SE_\lambda(\lambda z) = \lambda \boldsymbol{e}^{\lambda z} = \lambda \exp_{(\boldsymbol{e}^\lambda)}(z) \tag{5.26}$$

$$SE_\lambda^{\circ 2}(\lambda z) = \lambda \boldsymbol{e}^{\lambda \boldsymbol{e}^{\lambda z}} = \lambda \exp_{(\boldsymbol{e}^\lambda)}^{\circ 2}(z) \tag{5.27}$$

$$SE_\lambda^{\circ 3}(\lambda z) = \lambda \boldsymbol{e}^{\lambda \boldsymbol{e}^{\lambda \boldsymbol{e}^{\lambda z}}} = \lambda \exp_{(\boldsymbol{e}^\lambda)}^{\circ 3}(z) \tag{5.28}$$

and since this pattern obviously continues indefinitely, we find the relationship:

$$\boxed{SE_\lambda^{\circ n}(\lambda z) = \lambda \exp_{(\boldsymbol{e}^\lambda)}^{\circ n}(z)} \tag{5.29}$$

for all integer n, which allows theorems about one to be converted into the other.

5.3   Special cases of tetration

5.3.1   Half-exponential

Definition 5.30. The **half-exponential** is $\exp_b^{\circ 1/2}(x)$.

TODO

5.3.2 Self-power

Definition 5.31. The **self-power** is $SP(x) := x^x = {}^2x$.

Figure 5.7: Plot of x^x (**invertible**, real, imaginary)

Although this is a simple case of tetration, it is non-trivial since the immediate answer most would give for a power series expansion of ${}^2x = x^x$ is the exponential expansion $x^x = e^{x \log(x)} = \sum_k \frac{x^k \log(x)^k}{k!}$ which is not a Taylor series in x. Below are the Puiseux and Taylor series in x for this special case. They can be found through differentiation of x^x or e^{xe^x}, and the closed-forms for the exponential coefficients can be found in Sloane's OEIS [?]. [how its related to polygon notation]

A Puiseux series expansion:

Theorem 5.32.

$$
{}^2x = \sum_{k=0}^{\infty} \frac{\log(x)^k}{k!} \sum_{j=1}^{k} \binom{k}{j} j^{(k-j)} \tag{5.33}
$$

A Taylor series expansion:

Theorem 5.34 (Jovovic [A005727]).

$$
{}^2x = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \sum_{i=0}^{j} \binom{j}{i} i^{(j-i)} \tag{5.35}
$$

where $0^0 = 1$.

Iterated Second Hyperpowers We begin with the integer iterates:

$$SP^{\circ 1}(x) = x^x = \mathrm{T}(x, x, 1) \tag{5.36}$$

$$SP^{\circ 2}(x) = (x^x)^{(x^x)} = \mathrm{T}(x, x, 1 + x) \tag{5.37}$$

$$SP^{\circ 3}(x) = (x^x)^{(x^x)(x^x)(x^x)} = \mathrm{T}(x, x, 1 + x + x^{1+x}) \tag{5.38}$$

this leads us to notice that the third tower element is a sum of terms that satisfy the recurrence equation: $c_{n+1}(x) = c_n(x) + x^{c_n(x)}$ where $c_1(x) = 1$. which allows the iterate family of Frappier exponentials to be written: $FE^{[n]}(x) = \mathrm{T}\left(x, x, c_n(x)\right)$, however, this affords very little insight into the flow of Frappier exponentials.

two methods that work: * Lagrange interpolation of iterate-derivative matrix about $x = 1$ (only works because parabolic) * non-integer matrix powers of Bell matrix of $SH(x + 1) - 1$

$$\begin{aligned}
SP^{\circ t}(x) = {}& 1 + (x - 1) + t(x - 1)^2 \\
& + \left(-\frac{1}{2}t + t^2\right)(x - 1)^3 \\
& + \left(\frac{7}{12}t - \frac{5}{4}t^2 + t^3\right)(x - 1)^4 \\
& + \left(-\frac{7}{8}t + \frac{17}{8}t^2 - \frac{13}{6}t^3 + t^4\right)(x - 1)^5 \\
& + \left(\frac{169}{120}t - \frac{49}{12}t^2 + \frac{119}{24}t^3 - \frac{77}{24}t^4 + t^5\right)(x - 1)^6 + \cdots
\end{aligned} \tag{5.39}$$

5.3.3 Self-root

Definition 5.40. The **self-root** is $x^{1/x}$.

Theorem 5.41 (Jovovic [A008405])**.**

$$x^{1/x} = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \sum_{i=0}^{j} \binom{j}{i} (-i)^{(j-i)} \tag{5.42}$$

43

Figure 5.8: Plot of $x^{1/x}$ (**converge**, real, imaginary)

Theorem 5.43 (Jovovic [A003725]).

$$x^{1/x} = \sum_{k=0}^{\infty} \frac{\log(x)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (-j)^{(k-j)} \tag{5.44}$$

Theorem 5.45 ([A081048]).

$$\log(x^{1/x}) = \sum_{k=0}^{\infty} (x-1)^k (-1)^{k+1} \sum_{j=1}^{k} \frac{1}{j} \tag{5.46}$$

5.3.4 Super-sqrt (ssqrt)

Definition 5.47. The **super-sqrt** is $\mathrm{ssqrt}_k(x) = \exp\left(W_k(\log(x))\right)$.

Theorem 5.48 (Jovovic [A120980]).

$$\sqrt[2]{x}_4 = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (1-j)^{(j-1)} \tag{5.49}$$

Theorem 5.50 (not double-checked).

$$\log(\sqrt[2]{x}_4) = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \left(\frac{(-1)^k}{k+1} \right) \sum_{j=1}^{k+1} \left| s_j^{(k+1)} \right| j^{(j-1)} \tag{5.51}$$

44

Figure 5.9: Plot of ssqrt(x) (**invertible**, real, imaginary)

A general derivative of ssqrt(x) is:

$$\text{ssqrt}'(x) = \frac{\text{ssqrt}(x)}{x(\log(x) + \text{ssqrt}(x))} \tag{5.52}$$

The second super-root $\sqrt[2]{4}{z} = x$ solves the equation $x^x = z$ for x. An obvious alternative to using a function to solve this example equation is by using iteration. Before we can use iteration we must take the x-th root of both sides to get $x = z^{1/x}$ and take the reciprocal of both sides to get $1/x = (1/z)^{1/x}$. By replacing $1/x$ with $(1/z)^{1/x}$ repeatedly, we obtain an infinitely iterated exponential $1/x = {}^{\infty}(1/z)$. Now that there is only one x in the expression we can take the reciprocal again to find the relationship between the second super-root and infinitely iterated exponentials:

$$\sqrt[2]{4}{z} = x = \frac{1}{{}^{\infty}(1/z)} \tag{5.53}$$

A similar relationship can be found with the Lambert W-function. Starting with the definition of the second super-root we have already obtained $x = z^{1/x}$, and dividing by x gives $1 = (1/x)z^{1/x}$ so we can manipulate the equation into the form $we^w = z$ used by the definition of the Lambert W-function. Substitute $z = e^{\log(z)}$ to get $1 = (1/x)e^{\log(z)/x}$, and multiply by $\log(z)$ to get $\log(z) = (\log(z)/x)e^{\log(z)/x}$, and since the equation is of the form of the definition of the Lambert W-function, we

45

have $W(\log(z)) = \log(z)/x$. This involves only one x so solving gives the relationship between the second super-root and the Lambert W-function:

$$\sqrt[2]{z}_4 = x = \frac{\log(z)}{W(\log(z))} \tag{5.54}$$

At this point one could note that these techniques will only work for the second super-root, and not the third or fourth, because iteration would be so much more complicated in these and other cases. There is, however, one other super-root that can be simplified like the second super-root can; the infinite super-root. The infinite super-root is defined as $\mathrm{srt}_\infty(z) = x$ if and only if $^\infty x = z$ but we already know that the inverse function is $x = z^{1/z} = \sqrt[z]{z}$ by definition. Interestingly, this can be manipulated into a form that looks very similar to (5.53), by taking the reciprocal of the reciprocal an equivalent expression is obtained:

$$\mathrm{srt}_\infty(z) = x = \sqrt[z]{z} = \frac{1}{^2(1/z)} \tag{5.55}$$

Our aim is to understand the many relationships that exist between infinitely iterated exponentials, the second super-root, and the Lambert W-function, one of which shows that the first two are *strangely commutative* with respect to 2 and ∞:

$$\boxed{\begin{aligned} \mathrm{srt}_2(z) &= 1/^\infty(1/z) \\ \mathrm{srt}_\infty(z) &= 1/^2(1/z) \end{aligned}}$$

For more information see the section on topological conjugacy.

5.4 Miscellaneous functions

5.4.1 Exponential commutator

Definition 5.56. The **exponential commutator** is defined as:

$$EC(x) = y \text{ where } x^y = y^x \text{ and } x \neq y \tag{5.57}$$

The exponential commutator is one of the functions Knoebel [?] discusses in his review of iterated exponentials.

This can be found for $x \in \mathbb{R}$ by:

$$EC(x) = \begin{cases} H_1(x^{1/x}) & \text{if } x < \boldsymbol{e}, \\ H(x^{1/x}) & \text{if } x > \boldsymbol{e}. \end{cases} \tag{5.58}$$

it is debatable whether or not $EC(\boldsymbol{e}) = \boldsymbol{e}$ since this would not satisfy (1).

46

Question 5.59. *When does $x^y = y^x$?*

5.4.2 Exponential factorial

Definition 5.60. The **exponential factorial** is $EF(x) = x^{EF(x-1)}$ where $EF(1) = 1$.

Other initial conditions such as $EF(0) = 0$ have also been used before [?].
The exponential factorial may be written in tower notation as:

$$EF(x) = \prod_{k=0}^{\infty} (x - k) \qquad (5.61)$$

Using this it is interesting to approximate the extension of the exponential factorial to the real numbers using this, where the tower is "chopped off" at some point, for example:

$$EF(x) \approx \prod_{k=0}^{10} (x - k) \qquad (5.62)$$

or

$$EF(x) \approx \begin{cases} \log_{x+1}(EF(x+1)) & \text{if } x < 1 \\ f(x) & \text{if } 1 \le x \le 2 \\ x^{EF(x-1)} & \text{if } x > 2 \end{cases} \qquad (5.63)$$

where $f(x)$ is an approximation such as:

$$f(x) \approx 1 + 0.575571(x - 1) + 0.229718(x - 1)^2 + 0.0785761(x - 1)^3 \qquad (5.64)$$
$$f(x) \approx 1 + 0.570807(x - 1) + 0.232292(x - 1)^2 \qquad (5.65)$$
$$+ 0.114153(x - 1)^3 + 0.0234736(x - 1)^4$$

Using these approximations, or any for that matter, the 3-cycle that dominates the infinite tetration fractal[1], is somewhat apparent in various graphs of EF.

---

[1]see here

47

5.5 Topological Conjugacy

5.5.1 Exponential-like conjugates

Exponential-like conjugates satisfy the following diagram:

$$
\begin{array}{ccc}
\log_b(z) & \xrightarrow{\quad b^x = (h^{1/h})^x \quad} & (z) \\
\end{array}
$$

where the diagram has nodes $\log_b(z)$, (z), $\mathbb{C}$, $\mathbb{C}$, $\mathbb{C}$, $\mathbb{C}$ connected by arrows labeled $b^x = (h^{1/h})^x$, $h(x+1)$, $h(x+1)$, λx, λx, $h^x - 1$, $\lambda h(x+1)$, $\lambda h(x+1)$, $\lambda e^x = \log(b)e^x$.

where $b = e^\lambda = h^{1/h}$ and $h = H(b) = {}^\infty b$ and $\lambda = \log(b)$.

5.5.2 Lambert-like conjugates

Lambert-like conjugates satisfy the following diagram:

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\quad x^x \quad} & \mathbb{C} \\
\end{array}
$$
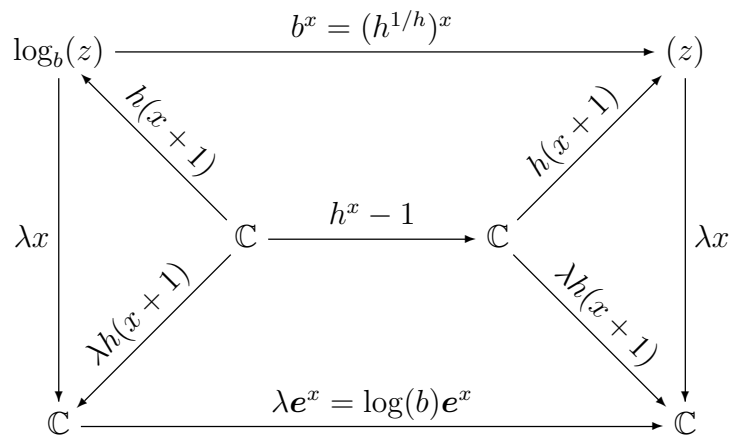
with nodes $\mathbb{C}$, $\mathbb{C}$, $W(z)$, (z), $\mathbb{C}$, $\mathbb{C}$ connected by arrows labeled x^x, e^x, e^x, $\frac{1}{x}$, $\frac{1}{x}$, $x e^x$, e^{-x}, e^{-x}, $x^{1/x}$.
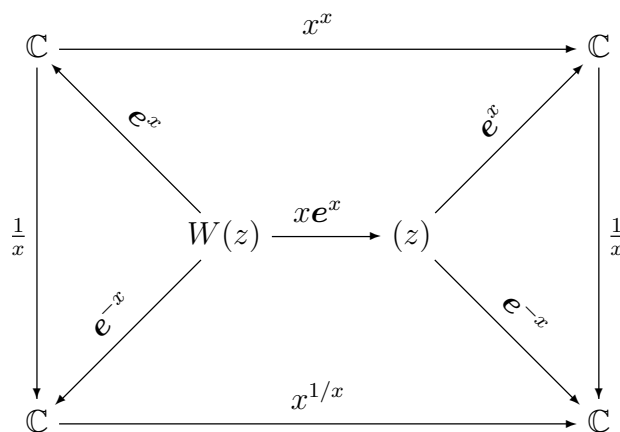
Appendix A

Function symbols

Symbol	Function Name
AS	Helm's alternating tetration series
BA	Bowers' array notation
CA	Conway's chained arrow notation
CN	Nelson's hyper-logarithm transform
DE	decremented exponentials $(\boldsymbol{e}^x - 1)$
EC	exponential commutator $(x^y = y^x)$
EF	exponential factorial
EG	exponential gamma function
H	Knoebel H function, infinitely iterated exponential
HC	critical piece of hyper-operations over real hyper-exponents
HD	critical piece of hyper-operations over real ranks
HE	half-exponential function
HF	Frappier hyper-operations
HR	Robbins hyper-operations
HT	Trappmann hyper-operations
HW	Galidakis HW function, hyper-product-logarithms
IG	Ioannis Galidakis' extension of tetration
MC	Munafo class function
PE	product exponentials $(x\boldsymbol{e}^x)$
RM	Robert Munafo's extension of tetration
SE	Devaney's scaled exponentials $(c\boldsymbol{e}^x)$
SM	Steinhaus-Moser polygon notation
SP	self-power function, second tetrate (x^x)
SR	self-root function, infinite super-root $(x^{1/x})$
W	Lambert W function, base-e product-logarithm

Appendix B

Tables of values

In the table below, $(\mathrm{BC_2})$ is used because of 3×3 matrix power rules in some CASs, $(\mathrm{DG_3})$ is used because Geisler gives terms 0-3 explicitly in [?], for (SW), w has been chosen such that $^{(-1/2)}\boldsymbol{e} = 1/2$, and average values are used for (CN^{-1}).

Table 2: $^{y}\boldsymbol{e}$ for $y \in [-1, 1]$.

y	Lin	$\mathrm{BC_2}$	$\mathrm{DG_5}$	IG	RM	SW	AR_{10}^{-1}	CN_{10}^{-1}
-1	0	**0**	$0.241 + 0.326\boldsymbol{i}$				0	
-0.9	0.1	**0.1**055	$0.394 + 0.170\boldsymbol{i}$				0.1062	
-0.8	0.2	**0.2**072	$0.475 + 0.034\boldsymbol{i}$				0.2079	
-0.7	0.3	**0.3**063	$0.508 - 0.060\boldsymbol{i}$				0.3063	
-0.6	0.4	**0.4**036	$0.524 - 0.105\boldsymbol{i}$				0.4029	
-0.5	0.5	**0.5**000	$0.547 - 0.106\boldsymbol{i}$				0.4986	
-0.4	0.6	**0.5**964	$\mathbf{0.5}93 - 0.081\boldsymbol{i}$				0.5946	
-0.3	0.7	**0.6**937	$\mathbf{0.6}67 - 0.046\boldsymbol{i}$				0.6917	
-0.2	0.8	**0.7**928	$\mathbf{0.7}66 - 0.018\boldsymbol{i}$				0.7910	
-0.1	0.9	**0.8**945	$\mathbf{0.8}80 - 0.002\boldsymbol{i}$				0.8934	
0	1	1	1				1	
0.1	1.1052	**1.1**100	$\mathbf{1.1}19 - 0.007\boldsymbol{i}$	0.0000	0.0000	0.0000	1.1121	0.0000
0.2	1.2214	**1.2**258	$\mathbf{1.2}37 - 0.016\boldsymbol{i}$				1.2310	
0.3	1.3499	**1.3**483	$\mathbf{1.3}56 - 0.022\boldsymbol{i}$				1.3584	
0.4	1.4918	**1.4**786	$\mathbf{1.4}84 - 0.021\boldsymbol{i}$				1.4961	
0.5	1.6487	**1.6**180	$\mathbf{1.6}26 - 0.015\boldsymbol{i}$				1.6465	
0.6	1.8221	1.7678	$\mathbf{1.7}89 - 0.006\boldsymbol{i}$				1.8123	
0.7	2.0138	1.9293	$\mathbf{1.9}77 + 0.003\boldsymbol{i}$				1.9971	
0.8	2.2255	2.1040	$\mathbf{2.1}91 + 0.008\boldsymbol{i}$				2.2055	
0.9	2.4596	2.2937	$\mathbf{2.4}32 + 0.010\boldsymbol{i}$				2.4433	
1	$\boldsymbol{e}$	2.5	$2.702 + 0.017\boldsymbol{i}$				$\boldsymbol{e}$	?

Appendix C

Series expansions

Exponential Coefficient Tables A listing follows of the first 10 (or so) exponential coefficients of tetration functions:

Table 3: A_{nk} where ${}^n x = \sum_{k=0}^{\infty} \frac{A_{nk} \log(x)^k}{k!}$

n,k	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	3	10	41	196	1057	6322	41393	293608	2237921
3	1	1	3	16	101	756	6607	65794	733833	9046648	121961051
4	1	1	3	16	125	1176	12847	160504	2261289	35464816	612419291
5	1	1	3	16	125	1296	16087	229384	3687609	66025360	1303751051
6	1	1	3	16	125	1296	16807	257104	4480569	87238720	1874561291
7	1	1	3	16	125	1296	16807	262144	4742649	96915520	2197675691
8	1	1	3	16	125	1296	16807	262144	4782969	99637120	2323474091
9	1	1	3	16	125	1296	16807	262144	4782969	100000000	2354318891
∞	1	1	3	16	125	1296	16807	262144	4782969	100000000	2357947691

Table 8: A_{nk} where ${}^n(x;z) = \sum_{k=0}^{\infty} \frac{A_{nk} \log(x)^k}{k!}$

n,k	0	1	2	3	4	5
1	1	z	z^2	z^3	z^4	z^5
2	1	1	$1+2z$	$1+6z+3z^2$	$1+12z+24z^2+4z^3$	$1+20z+90z^2+80z^3+5z^4$
3	1	1	3	$10+6z$	$41+48z+12z^2$	$196+360z+180z^2+20z^3$
4	1	1	3	16	$101+24z$	$756+360z+60z^2$
5	1	1	3	16	125	$1176+120z$
6	1	1	3	16	125	1296
∞	1	1	3	16	125	1296

Table 4: A_{nk} where $^n x = \sum_{k=0}^{\infty} \frac{A_{nk}(x-1)^k}{k!}$

n,k	0	1	2	3	4	5	6	7	8	9	10
1	1	1	0	0	0	0	0	0	0	0	0
2	1	1	2	3	8	10	54	-42	944	-5112	47160
3	1	1	2	9	32	180	954	6524	45016	360144	3023640
4	1	1	2	9	56	360	2934	26054	269128	3010680	37616880
5	1	1	2	9	56	480	4374	47894	574888	7829424	116392080
6	1	1	2	9	56	480	5094	60494	823528	12365424	206078880
7	1	1	2	9	56	480	5094	65534	944488	15359184	274118880
8	1	1	2	9	56	480	5094	65534	984808	16629264	312523680
9	1	1	2	9	56	480	5094	65534	984808	16992144	327038880
∞	1	1	2	9	56	480	5094	65534	984808	16992144	330667680

Table 8: A_{nk} where $^n(x;z) = \sum_{k=0}^{\infty} \frac{A_{nk}(x-1)^k}{k!}$

n,k	0	1	2	3	4	5
1	1	z	$-z+z^2$	$2z-3z^2+z^3$	$-6z+11z^2-6z^3+z^4$	$24z-50z^2+35z^3-10z^4+z^5$
2	1	1	$2z$	$3z^2$	$-2z+6z^2+4z^3$	$10z-45z^2+40z^3+5z^4$
3	1	1	2	$3+6z$	$8+12z+12z^2$	$10+90z+60z^2+20z^3$
4	1	1	2	9	$32+24z$	$180+120z+60z^2$
5	1	1	2	9	56	$360+120z$
6	1	1	2	9	56	480
∞	1	1	2	9	56	480

Table 5: A_{nk} where $\log(^n x) = \sum_{k=0}^{\infty} \frac{A_{nk}(x-1)^k}{(k+1)!}$

n,k	0	1	2	3	4	5	6	7	8	9
2	1	1	-1	2	-6	24	-120	720	-5040	40320
3	1	1	5	2	44	-96	986	-6056	52272	-466920
4	1	1	5	26	104	804	4136	38296	247320	2586000
5	1	1	5	26	224	1524	15896	149176	1830384	21682560
6	1	1	5	26	224	2244	23456	297016	3947184	60263760
7	1	1	5	26	224	2244	28496	377656	5852304	96551760
8	1	1	5	26	224	2244	28496	417976	6759504	122255760
9	1	1	5	26	224	2244	28496	417976	7122384	133142160
∞	1	1	5	26	224	2244	28496	417976	7122384	136770960

And a listing of the first 10 (or so) exponential coefficients of super-root functions:

Table 6: A_{nk} where $\mathrm{srt}_n(z) = \sum_{k=0}^{\infty} \frac{A_{nk}\log(z)^k}{k!}$

n,k	0	1	2	3	4	5	6	7	8	9	10
2	1	1	-1	4	-27	256	-3125	46656	-823543	16777216	-387420489
3	1	1	-1	-2	33	-184	-695	32124	-369215	-1298816	143686161
4	1	1	-1	-2	9	116	-1175	-3786	92449	1565416	-41559759
5	1	1	-1	-2	9	-4	625	-7146	-33551	821512	-2333439
6	1	1	-1	-2	9	-4	-95	5454	-60431	-312488	8371521
7	1	1	-1	-2	9	-4	-95	414	40369	-554408	-2968479
8	1	1	-1	-2	9	-4	-95	414	49	352792	-5387679
9	1	1	-1	-2	9	-4	-95	414	49	-10088	3684321
∞	1	1	-1	-2	9	-4	-95	414	49	-10088	55521

Table 7: A_{nk} where $\mathrm{srt}_n(z) = \sum_{k=0}^{\infty} \frac{A_{nk}(z-1)^k}{k!}$

n,k	0	1	2	3	4	5	6	7	8	9	10
2	1	1	-2	9	-68	740	-10554	185906	-3891320	94259952	-2592071760
3	1	1	-2	3	28	-510	4926	-10500	-900304	26247816	-392237520
4	1	1	-2	3	4	30	-2094	33810	-338176	3412080	-93913560
5	1	1	-2	3	4	-90	1506	-28350	444704	-5489064	47130840
6	1	1	-2	3	4	-90	786	-630	-166816	4490136	-79146360
7	1	1	-2	3	4	-90	786	-5670	75104	-2132424	55724040
8	1	1	-2	3	4	-90	786	-5670	34784	226296	-22597560
9	1	1	-2	3	4	-90	786	-5670	34784	-136584	2804040
∞	1	1	-2	3	4	-90	786	-5670	34784	-136584	-824760

Table 8: A_{nk} where $\log(\mathrm{srt}_n(x)) = \sum_{k=0}^{\infty} \frac{A_{nk}(x-1)^k}{(k+1)!}$

n,k	0	1	2	3	4	5	6	7	8	9
2	1	-3	17	-146	1704	-25284	456224	-9702776	237711888	-6593032560
3	1	-3	11	-26	-266	6336	-73662	166608	18635328	-604483080
4	1	-3	11	-50	394	-5364	82788	-1262064	20814120	-426484080
5	1	-3	11	-50	274	-1044	-14652	562416	-12551184	242958960
6	1	-3	11	-50	274	-1764	18108	-351504	8193456	-184206240
7	1	-3	11	-50	274	-1764	13068	-69264	-1332144	73136160
8	1	-3	11	-50	274	-1764	13068	-109584	1389456	-36030240
9	1	-3	11	-50	274	-1764	13068	-109584	1026576	-6999840
∞	1	-3	11	-50	274	-1764	13068	-109584	1026576	-10628640

Appendix D

Credits

TODO

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