Terminology Proposals for a Hyper operation Environment

Hello everybody! I am pleased to be involved in this Forum from its beginning, since I am convinced that the subject it covers will surely become of great importance in the future developments of mathematics. Nevertheless, I should like to propose that the Forum adopts the simplest and clearest approach, for involving the maximum number of people, while keeping the necessary strict rigorous mathematical formalization to a second phase, or to specialized sub-forums. Clear and practical terminological choices are therefore indispensable. Here are my proposals, as a result of a long cooperation with Konstantin Rubtsov (KAR). These conventions were in fact used in a report that we jointly prepared in 2003 ("published" on the Web on 17-09-2004) and mentioned in the bibliography of the Stephen Wolfram's book "*A New Kind of Science*". See, for instance:

http://forum.wolframscience.com/showthread.php?s=a62c97b0b7fa64d17ffee94adf2eb8b7&threadid=579
http://forum.wolframscience.com/showthread.php?s=a62c97b0b7fa64d17ffee94adf2eb8b7&threadid=956

In fact, in an extremely simple, informal and intuitive way, we can define a particular ternary operation, i.e. involving three numbers, as follows (the operation becomes binary for any constant *s*):

$$H(a,b) \,|\, s = c$$

with:
- a: the first operand, or the *base*
- b: the second operand, or the *exponent* or *level* or *degree*
- s: the (ternary operation) *rank*

such that: $H(a,a) \,|\, s = H(a,2) \,|\, (s+1)$

and: $H(a, H(a,a) \,|\, s) \,|\, s = H(a,3) \,|\, (s+1)$

By using an infixed operator, we could, more easily and clearly, write:

$$a \boxed{s}\, b = c$$
$$a \boxed{s}\, a = a \boxed{s+1}\, 2$$
$$a \boxed{s} \left(a \boxed{s}\, a\right) = a \boxed{s+1}\, 3$$

and:
$$\underbrace{a \boxed{s} \left(a \boxed{s} \left(a \boxed{s} \ldots\ldots \left(a \boxed{s}\, a\right) \ldots\ldots\right)\right)}_{n \text{ times}} = a \boxed{s+1}\, n$$

We see that any operation of rank *s*, applied on its *base* and iterated *n* times, gives an operation of rank *s* + 1, with *exponent* (*level*, *degree*) equal to *n*. It can be shown that this ternary operation can be put in bijection with the elements of the Ackermann function, if we put $a \boxed{1}\, b = a \boxed{+}\, b = a + b$. In this case, we obtain an infinite hierarchy of operations (called the *Grzegorczyk Hiierarchy*), three of which are the well known binary operation of classical algebra. In fact, we get:

s=1	$a \boxed{1}\, b = a + b$	*addition*
s=2	$a \boxed{2}\, b = a \times b = a * b = a \cdot b = ab$	*multiplication*
s=3	$a \boxed{3}\, b = a \wedge b = a^b$	*exponentiation*

Indeed, we can verify that:

$$\underbrace{a + \left(a + \left(a + \ldots \left(a + a\right) \ldots\right)\right)}_{n} = a \cdot n \qquad \textit{addition} \to \textit{multiplication}$$

$$\underbrace{a \cdot \left(a \cdot \left(a \cdot \ldots \left(a \cdot a\right) \ldots\right)\right)}_{n} = a \wedge n = a^n \qquad \textit{multiplication} \to \textit{exponentiation}$$

$$\underbrace{a \wedge \left(a \wedge \left(a \wedge \ldots \left(a \wedge a\right) \ldots\right)\right)}_{n} = a \# n = {}^{n}a \qquad \textit{exponentiation} \to \textit{tetration}$$

The parentheses are not necessary in the first two lines, because the operations involved are both commutative and associative. The third line is an implicit definition of ***tetration***, the famous fourth operation of the hierarchy, to be called the ***hyperoperation** hierarchy*. These definitions imply a systematic bracketing convention that we might call "*priority to the right*".

It is then proposed to keep the name *hyperoperation(s)* for indicating each and all individual member(s) of such hierarchy, for every possible rank. Prefix ***hyper-*** would then refer to any generic member, unless a particular *hyperoperation rank* is specified (e.g.: the *hyper-4* or *tetra-*, and *hyper-5* or *penta-* operations).

It is also proposed to reserve prefix ***super-*** for another precise and specific meaning (see hereafter).

Coming back to a generic *hyperoperation* of rank *s*, to be written as follows:

$$a \; \boxed{s} \; b = c$$

we can define the following two inverse operations:

- if we know *b* and *c*, base *a* is given by the *hyperroot* (level *b*) of *c*, or the *b-th hyperrot* of *c*;
- if *a* and *c* are known, exponent *b* is given by the *hyperlog* (to the base *a*) of *c*.

These couple of generic ***inverse hyperoperations*** can be symbolically shown by using two logotypes visually recalling the standard root and logarithm symbols, as follows:

$$a = {}^{b}\overset{s}{\ulcorner} c \qquad \text{the } b\text{-th } \boldsymbol{s}\textbf{-hyperroot} \text{ of } c$$

$$b = {}_{a}\underset{s}{\llcorner} c \qquad \text{the } \boldsymbol{s}\textbf{-hyperlog} \text{ of } c, \textit{ to the base } a$$

As we know, the implementation of these definitions, for ranks 1 and 2, gives a trivial result (the *s-hperroot* and *s-hyperlog* inverse operations, for ranks *s*=1 and *s*=2, are coincident). In fact:

for *s*=1	$a+b=c$	(commutative)	→	$a=c-b$	$b=c-a$	*subtraction*
for *s*=2	$a \cdot b = c$	(commutative)	→	$a=c/b$	$b=c/a$	*division*

The same definitions, for ranks *s*=3 and *s*=4, give the following results:

for *s*=3	$a \wedge b = c$	(non-commutative)	→	$a = {}^{b}\overset{3}{\ulcorner} c = \sqrt[b]{c}$	*root*
				$b = {}_{a}\underset{3}{\llcorner} c = \log_a c$	*logarithm*
for *s*=4	$a \# b = c$	(non-commutative)	→	$a = {}^{b}\overset{4}{\ulcorner} c = {}^{b}\big)\overline{c}$	***superroot***
				$b = {}_{a}\underset{4}{\llcorner} c = \mathrm{slog}_a c$	***superlog***

As we see, the convention stipulates that the name of a particular *hyperoperation* (and of its inverses) can be obtained by using the prefix *super-*, applied to the names of the homologue operation of the immediate lower rank. In this case, *tetration* could also be called the *super-exponentiation* or *superpower* and, therefore, its two inverses *superroot* and *superlog*, without any possible ambiguity. *Pentation*, for example, could also be called *supertetration* or *supertower*. Moreover, as indicated in one of the above expressions, for the study of *tetration*, it would be appropriate to agree on a conventional logotype for the *superroot*.

These conventions can be applied to other names used in this field (with some unused extensions):

s=1	$a+b=c$	*addition*	*sum*
s=2	$a \cdot b = c$	*multiplication*	*product – supersum (?)*
s=3	$a \wedge b = c$	*exponentiation*	*power – superproduct (?)*
s=4	$a \# b = c$	*tetration*	*tower – superpower*

Actually, the terminology of the third column reflects the abstract names of the operations, while that of the fourth column should indicate the results obtained out of the same operations (but this is not always generally accepted). In fact, with this convention, prefix *super-*, as applied to the name of a particular hyperoperation, would indicate the hyperoperation of the immediate higher rank and, more precisely, an iteration of the lower rank operation.

This convention is tacitly and practically adopted in the symbol choices used by some existing pocket calculators, through the repetition of an operation symbol for indicating the higher rank operation. For instance, by also including some hypothetical and unused examples, we could write:

$a+b$						
$a++b$	$=$	$a*b$				
$a+++b$	$=$	$a**b$	$=$	$a\wedge b$		
$a++++b$	$=$	$a***b$	$=$	$a\wedge\wedge b$	$=$	$a\ \#\ b$

In these examples, $a**b$ would indicate the *superproduct* (via an *iterated multiplication*), i.e. *exponentiation*, and $a\wedge\wedge b$ would stand for the *superpower* (via an *iterated exponentiation*), i.e. *tetration*. This convention is not self-contradictory and can be extended. However, for practical reasons, a standard sequential graphic operator (e.g. by the ASKII coding standards) must be chosen at least for ranks 1 to 4. For *pentation*, we shall see ... later.

The extension to higher ranks of convention $a\wedge\wedge b$ for indicating *tetration* corresponds to the known "arrow convention". Personally, I prefer not to recommend its extension to higher ranks, because it might be a source of confusion (e.i. *pentation* would be $a\wedge\wedge\wedge\ b$!?!). Moreover, there is no reason for attributing to *exponentiation* (*power*) a central role in the hierarchy (one arrow). If we don't have a graphic operator for *pentation* (we might choose "$", for example), we could write "##" (*supertower*) or "+++++". If need be, the convention of the first column could be more acceptable. The denomination "power-tower" is indeed to be banned, because it is the source of a total confusion.

Concerning *tetration*, disregarding for the moment the graphical compact symbols shown below, the following terminological conventions can also be adopted (extendable to all the ranks $s > 4$):

$$a\ \#\ b = {}^{b}a = a\text{-tetra-}b, \quad a = \sqrt[b]{c} = b^{\text{th}}\text{srt } c = \text{tetraroot}^{b}\, c\,, \quad b = \text{slog}_a\, c = \text{tetralog}_a\, c$$

As far as the graphical symbols are concerned, a more compact notation has already been widely adopted for ranks 2, 3 and 4. For instance, as it is normally done, we can write (the last one being a recent convention):

$$\boxed{a\cdot b = ab \qquad\qquad a\wedge b = a^b \qquad\qquad a\ \#\ b = {}^{b}a}$$

I think that this scheme is quite illogical, but extremely clear, nice and acceptable. However, this compact convention is limited to the ranks 2, 3 and 4. A very clear and global convention could be obtained by generalizing the first column of the upper table and by putting:

$$a\ \boxed{s}\ b = a\overset{s}{+}b \qquad \text{with } \overset{s}{+} = (+)^s \quad \text{i.e. } (+) \text{ repeated } s \text{ times}$$

This convention would indeed be applicable to all the hyperoperation hierarchy. Unfortunately, it would be too complicated in the implementations and, graphically, not more compact than the $\boxed{s}$ operator itself. Finally, it is interesting to observe that the following formulas are valid:

$$a\boxed{3}a = a\wedge a = a\ \#\ 2$$

$$a\boxed{2}a = a\cdot\ a = a\wedge 2$$

$$a\boxed{1}a = a+a = a\cdot 2$$

This table suggests the possibility of the existence of a rank zero rank operation (z*eration*), lower than the rank of *addition*, satisfying the following relation (a binary extension of the "*successor*" monary operation):

$$a\boxed{0}a = a \circ a = a + 2$$

But I think that this new, very important and controversial issue should be analyzed separately.

Nevertheless, I am convinced that the *hyperoperation hierarchy* could be extendable (... at least) to all the relative integer values of the ranks (even below 0!) and that, therefore, <u>it does not admit any initial step</u> (a *don't move* operation, for example). But, also this can be an interesting subject of discussion.

Coming back to *tetration* (s=4), it is useful to compare it with *exponentiation* (s=3), by supposing that ***x*** is the variable base and ***n*** the constant exponent:

$y = x^n$ <u>*x*-power-*n*</u>	$y = {}^n x$ <u>*x*-tower-*n*</u>
*n-degree **power** function*	*n-level **tower** function*
(exponentiation)	*(tetration)*

On the contrary, in case we suppose ***b*** to be the constant base and ***x*** the variable exponent, we shall have:

$y = b^x$ <u>*b*-exp-*x*</u>	$y = {}^x b$ <u>*b*-tetra-*x*</u>
*base-b **exponential** function*	*base-b **tetrational** function*
(exponentiation)	*(tetration)*

I suppose that these proposals are more or less in line with what has already been adopted by some of the Forum's participants like, for instance by Andrew Robbins. The systematic choice of the logic "*priority to the right*" parentheses (bracketing) convention for the definition of the hyperoperation hierarchy is compulsory because the complementary convention does not seem to be appropriate, since it does not generate elementary operation. In fact, any left-priority tower will ... collapse and systematically produce three-level "inhomogeneous towers" e.g.: $\left(\left(\left(\left(a \wedge a\right) \wedge a\right) \wedge a\right) \wedge a\right) = a^{a^4}$. Moreover, I think that the logical right-priority convention was also chosen by Euler, in the framework of his famous study of the convergence of "*infinite towers*".

Please see also the annexed table with a "Terminology and Symbols Reminder" s used in the KAR/GFR threads of the NKS Forum.

Thank you for your kind attention .

GFR

NB: The present notes are prepared only for standardizing the nomenclature to be possibly applied in the Forum. The priority of use and authoring of the proposed terms should be the object of another thread, concerning the historical background of the mathematical research in this field.

Terminology and Symbols Reminder

From: http://forum.wolframscience.com/showthread.php?s=ec62eee7db84cd5ca2543ed3fe5a96f7&threadid=1372

$\boxed{s}$	- general hyper-operator of rank *s*, with $\boxed{1} = +$, $\boxed{2} = \times$, $\boxed{3} = \wedge$ and $\boxed{0} = \circ$ (zeration), $\boxed{4} = \#$ (tetration);
$y = {}^{z}x$	- tetration; *x* # *z*; *x*-tetra-*z* or *x*-tower-*z*, i.e. *x* raised to *x*, *z* times (*z* is the super-exponent);
$y = x \circ z$	- zeration; *x*-zerated-*z*, max(*x*,*z*)+1 if $x \neq z$, *x*+2 = *z*+2 if *x* = *z*;
$x = y \Delta z$	- deltation; *y*-delta-*z* (inverse function of $y = x \circ z$);
${}^{z}\lceil y^{s}$	- the z^{th} hyper-root of *y*, rank *s*; for s = 4 : ${}^{z}\lceil y^{4} = {}^{z}\rangle y$ (the z^{th} super-root or tetra-root of *y*) and, for s = 3: ${}^{z}\lceil y^{3} = \sqrt[z]{y}$ (the z^{th} root of *y*);
${}_{x}\lfloor y_{s}$	- the hyper-logarithm, base *x*, of *y*, rank *s*, that can also be written as: $\underset{s}{\text{hlog}}\ \underset{x}{}\ y$; for *s* = 4: ${}_{x}\lfloor y_{4} = \text{slog}_x y$ (super-log, or tetra-log, base *x*, of *y*) and for *s* = 3: ${}_{x}\lfloor y_{3} = \log_x y$;
$\rangle y$	- the square super-root (tetra-root) of *y* : $\rangle y = {}^{2}\lceil y^{4}$;
plog(*x*)	- the product logarithm (Lambert's Function), the inverse function *y* of $x = y \cdot e^{y}$;
ln *y*	- the natural logarithm of *y*, i.e. : $\log_e y = {}_{e}\lfloor y_{3}$;
$\text{slog}_a y$	- the super-logarithm (tetra-logarithm), to the base *a*, of *y*, i.e.: $\text{slog}_a y = {}_{a}\lfloor y_{4}$.
sln *y*	- the natural super-logarithm (natural tetra-logarithm) of *y*, i.e.: $\text{slog}_e y = {}_{e}\lfloor y_{4}$.
