# Neutral Fixed Points, Borel Sums, and Mellin Transforms

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**Abstract**

This paper is an analysis of neutral fixed points of holomorphic functions. We discuss the Mellin transformable nature of their iterates–including fractional iterates. We perform coded Borel sums. This paper will be a discussion of mapping Leau petals to a half-plane; and the Mellin transform's role in describing the iterates. We pay specifically close attention to the case $f(z) = e^z - 1$ and its fixed point at $z = 0$.

## 1 Introduction

This short paper was spawned from a series of conversations on the Tetration Forum [7] between a multitude of users. Primarily this work is my own, though developed through conversations with Gottfried Helms, and Henryk Trappman. And follows closely to research done by the two of them; which cumulated in to a vast resource of results–syphoned from scattered resources.

To begin the paper, we should start what began the train of thought for this venture into the study of neutral fixed points, and their dynamics. For that it is a debt we owe Gottfried Helms and a peculiar matrix algorithm. Consider the function $f(z) = e^z - 1$. This function has a neutral fixed point at zero, with multiplier 1. By this, we mean that:

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \end{aligned}$$

What a *neutral fixed point* entails; more generally, is that a function $g(z_0) = z_0$ while $|g'(z_0)| = 1$–where $z_0$ is the *neutral fixed point*. Where the term *multiplier* is to mean the value $g'(z_0)$. For a fixed point $z_0$ of $g$ to be neutral, we require that $|g'(z_0)| = 1$.

The heart of Gottfried Helms' work on the matter is a tad difficult to explain without entire context. But we can still give a good analysis of what the question is asking. Let's create the function $h$, such that:

$$h(h(z)) = f(z)$$

And let's ask that this function be holomorphic at $z = 0$. Well, unfortunately, this is impossible. There is no such holomorphic function at 0. This is a result by I. N. Baker [1]–but we reference Karlin and McGregor [3]. Baker's articles are almost exclusively German, but we can summarize that there exists no analytic semi-group $f^{\circ t}(z)$ for $z$ near a neutral fixed point–unless $f$ is a linear fractional transformation. Which means, of the form $f(z) = \frac{z}{cz+1}$. What Helms set out to do, instead, was to approximate said holomorphic function *if it were to exist*–by equating Taylor coefficients. By such, I invite the reader to follow with me in my quick bastardization of Helms' idea:

$$\begin{aligned} h(z) &= \sum_{k=1}^{N} h_k z^k + O(z^{N+1}) \\ h(h(z)) &= \sum_{k=1}^{N} h_k \left( \sum_{j=1}^{N} h_j z^j \right)^k + O(z^{N+1}) \\ &= \sum_{k=1}^{N} \frac{z^k}{k!} + O(z^{N+1}) \end{aligned}$$

The algorithm for this is always solvable; and we are able to extrapolate a sequence of numbers $\{h_k\}_{k=1}^{\infty} \in \mathbb{R}$. The true question that Helms asked is: *what is the growth of* $h_k$? Gottfried had a very strong asymptotic numerically which appeared more and more to be true. Which, Trappman in a more ambitious manner; confirmed the same asymptotic for other functions and other root functions.–Whereby root functions, we mean functions $\sqrt[k]{f} = a$ where $a^{\circ k} = f$.

The exact asymptotic is rather troublesome to write out, which I will not do. It is irrelevant to our discussion of Mellin transforms. All that's needed from the asymptotic (for our work here at least), is that $h_k = O(c^k k!)$. And this is in essence the basis of Helms' asymptotic, which was further confirmed by Trappman on different functions–and varying semi-iterates.

This result was found to be true, by words of Ecalle and much of his work–but also by his own comments on a similar problem. The function $h$ at 0 is of Gevrey class 1. This means that $h$ is not holomorphic in a neighborhood of zero (there is a branching problem at 0 itself, but it is holomorphic to, exclusively, the left or right of 0). But at 0 itself, it has very specific growth in its Taylor series–which confirm that $h_k = O(c^k k!)$–which is precisely the statement of Gevrey class 1. We reference [2, 6]–which although they are not the originators of these ideas–give a very encyclopedic description of this result and the adjacent theory.

As we continue deep into this work; it's important to identify some common themes before we get started. We will reserve the variable $f$ to mean the function of interest. We will always map the value zero to itself; where we assume this is the fixed point–$f(0) = 0$. We will then additionally ask that $f'(0) = e^{2\pi i q/m}$ for integers $q, m$; by which we can (through iteration) notice that $f^{\circ m}(z) = z + O(z^2)$–where now we can just pretty much always assume that $f'(0) = 1$; we are just working with an iterate.

For the cases where $f'(0) = e^{2\pi i \xi}$ where $\xi$ is irrational–we will spend a separate section on. When $\xi$ is irrational, the mapping $f$ is no longer parabolic and obeys a decent amount of differing rules (though still similar). So unless stated otherwise, we can assume that $f'(0) = 1$. Whereupon, as an example case, we will frequently return to $f(z) = e^z - 1$. Especially as this stands as an iconic example, with much explanatory literature.

The central goal of this short paper is to talk about applying a Mellin transform on the function $h(h(z)) = f(z)$. In many cases, this will appear as nonsense by first principles. But by performing the correct change of variables; it is possible.

## 2 Gevrey Class 1

Let's begin by taking a holomorphic function $f : \mathbb{C} \to \mathbb{C}$–where additionally $f(0) = 0$ and $f'(0) = 1$. As can be found in I. N. Baker's work; there can be no holomorphic function $h$ such that $h(h(z)) = f(z)$ unless $f$ is a linear fractional transformation/Möbius transformation. So unless $f(z) = \frac{z}{cz+1}$, there can be no half iterate holomorphic at 0. We have compiled two different references here , which [3] by Karlin and McGregor develops further results from an elusive paper by I. N. Baker (much of his work is untranslated from German); and said elusive paper [1].

The concept of the Gevrey class $m$ is meant to say, that a sequence of numbers $\{b_k\}_{k=1}^{\infty}$ satisfy:

$$|b_k| \leq c^k k!^{1/m}$$

For some number $c > 0$ and $m$ is a natural number. In order to understand the Gevrey class, it is important to start from the top of the pile. Where in, it is used to describe the coefficients of the Abel function. But in order to do this, we must first make a change of variables. This allows us to avoid Gevrey class $m$, and only talk of Gevrey class 1. For our interests, this suffices.

This change of variables appears in Ecalle's work, but we trace it solely back to John Milnor's work [4], and his discussion of parabolic iterations. By which we write:

$$f(z) = z\left(1 + az^{\ell} + O(z^{\ell+1})\right)$$

We will call the value $\ell + 1$ the *multiplicity* of the neutral fixed point–but we prefer the denomination $\ell$ for the *valit*; a term more accustomed to Trappman's work. For the function $f(z) = e^z - 1$, the valit is $\ell = 1$. According to the Leau-Fatou Flower theorem [4], there are precisely $2\ell$ petals about the value 0, which when unionized, make a neighborhood about 0 (Which is a rough statement of

The Leau-Fatou Flower Theorem). Additionally, $\ell$ of these petals are attracting and the others are repelling.

To be consistent we will call a petal $\mathcal{P}$. Where additionally $f : \mathcal{P} \to \mathcal{P}$ if it is an attracting petal, and $f^{-1} : \mathcal{P} \to \mathcal{P}$ if it is a repelling petal. For each petal there exists a characteristic vector $v = u + iv$. The attraction vectors are given as $\ell a v^\ell = -1$, and repulsion vectors are $\ell a v^\ell = 1$. (Milnor calls these the **attraction vectors**, and the **repulsion vectors**; and we borrow his jargon). Where each characteristic vector of every petal, represents a $2\ell$ root of unity.

For attracting petals, we can map $\mathcal{P} \to \Re(z) > K$ for large enough $K > 0$. For repelling petals we can map $\mathcal{P} \to \Re(z) < -K$, for $K$ similar. This change of variables will be our canonical use of the variable $F$. Let us write:

$$w = \varphi(z) = \frac{-1}{\ell a z^\ell}$$

Now let's write the inversion $z = \varphi^{-1}(w) = \dfrac{1}{\sqrt[\ell]{-\ell a w}}$; where within a sector of width $\pi/\ell$ this inversion is perfect (which is the maximal width of our petal near enough to 0). By which we can write $F(w)$ as:

$$F(w) = \varphi \circ f \circ \varphi^{-1}(w)$$

This makes a function $F$ which is holomorphic in the right half plane (left half plane if repelling)–so long as we choose far enough out. By, which, the function:

$$F(w) = w + 1 + o(1)$$

And so $F(w) : \{\Re(w) > K\} \to \{\Re(w) > K + 1\}$ if we are talking about an attracting petal; and $F(w) : \{\Re(w) < -K\} \to \{\Re(w) < 1 - K\}$ if we are talking about a repelling petal. We can focus entirely on attracting petals; because we can always make the change of variables to $f^{-1}$, which flips the attracting/repelling petals.

Gevrey class 1 makes its first appearance here. But in a difficult manner:

$$F(w) = w + 1 + \sum_{k=1}^{\infty} F_k w^{-k}$$

Where the sequence $|F_k| = O(d^k)$ for $d > 0$–because $F$ is meromorphic at $\infty$. This is to say that on the plane $\{\Re(w) > K\} \cup \infty$, our function $F$ takes this domain to itself–and fixes infinity from any path on this domain.

Which means this expression does converge, but rather far out if necessary (choose $K$ large enough). Despite this, the function $F$ is a holomorphic function, it is simply not *usually* expanded in this manner. The following result comes from the following two monoliths on Abel functions on $f$'s dynamics, and the relation to Borel Sums: [2, 6]. On this space we can introduce two concepts. The first is that:

$$A(F(w)) = A(w) + 1$$

Where $A$ is now expressed using the Gevrey Class 1 sequence structure:

$$A(w) = w + C\log(w) + \sum_{k=1}^{\infty} A_k w^{-k}$$

Where $A_k$ satisfies the appropriate bounds: $|A_k| \leq c^k k!$ for some $c > 0$. Moving forward, if we make a function $H(w) = \varphi \circ h \circ \varphi^{-1}(w)$, we just as well encounter a construction:

$$H(w) = w + \frac{1}{2} + \sum_{k=1}^{\infty} H_k w^{-k} = A^{-1}(A(w) + \frac{1}{2})$$

Where the solution:

$$H(H(w)) = F(w)$$

Is a coefficient check, based off of the same trick Helms' did in the beginning of this paper. All we must do is solve coefficient by coefficient–a manner Helms felt most natural with matrices. And from which, we have that $|H_k| \leq q^k k!$ for some $q > 0$.

This for the most part concludes what the reader needs to understand from outside this paper to do with Gevrey classes. As I will prove results relating to iterations, they will solely rely on the requirement of the growth of these coefficients. Where, beautifully, all we need is to understand that iterates of $F$ have an asymptotic expansion of Gevrey class 1.

## 3 What space means to us

We will now define the central space we care about. Let $P(w)$ be a holomorphic function for $\Re(w) > K$ (or $< -K$ if we're repelling). Then we define the Gevrey space as follows:

**Definition 3.1** (The Gevrey space)**.** The space $B \in \mathfrak{G}$ is the space of functions which have an expansion:

$$B(w) = P(w) + o(1)$$

For $B(w)$, $P(w)$ holomorphic for $\Re(w) > K$ ( $< -K$). While additionally, the term:

$$o(1) \sim \sum_{k=1}^{\infty} P_k w^{-k}$$

And,

$$P(w) = O(w)$$

Where $|P_k| = O(c^k k!)$ for some $c > 0$.

Clearly this is a space, $\beta, \gamma \in \mathbb{C}$ and $B(w), L(w) \in \mathfrak{G}$, then $\beta B(w) + \gamma L(w) \in \mathfrak{G}$–$0 \in \mathfrak{G}$ trivially. As such, we are trying to map the space $\mathfrak{G}$ through the Mellin transform. From here we achieve a potluck of tools.

We will draw most of our reference from Don Zagier [8], and his work on the Mellin transform (which is a more Analytic Number Theory, kind of thing). We can effectively begin the novel portions of this discussion by talking about Euler, and the early roots of Borel sums.

We close this section with a couple of identifications: we can always make the change of variables $B(w+K) - K$, so we can assume that $K = 0$. Additionally we can assume that $P_k = O(d^k k!)$–for $0 < d < 1$. This is because the addition $+K$ creates an exponential decay of $K^{-\delta k}$ for $\delta > 0$. Choosing $K$ large enough such that $c/K^\delta = d < 1$ suffices.

This can be better explained by returning to our change of variables at the beginning of the previous section. Instead of writing $w = \varphi(z) = \frac{-1}{\ell a z^\ell}$, and $z = \varphi^{-1}(w) = \frac{1}{\sqrt[\ell]{-\ell a w}}$; we write $w = \varphi_K(z) = \frac{-1}{\ell a z^\ell} - K$ and $z = \varphi_K^{-1}(w) = \frac{1}{\sqrt[\ell]{-\ell a (w+K)}}$. Then we'd write:

$$F(w) = \varphi_K \circ f \circ \varphi_K^{-1}(w)$$

This is a much more apt change of variables for our purposes. We will leave $K$ as a free variable, but there exists a large enough value for $K$ that will always work.

From here, we can restrict the Gevrey space $\mathfrak{G}$ to the space $\mathfrak{G}_0$ to mean that our object is holomorphic for at least $\Re(w) > 0$. Upon which, we allow for the first hint of Mellin transforms:

$$\sum_{n=0}^{\infty} B(n+1) \frac{(-x)^n}{n!} = \phi(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(w) B(1-w) x^{-w}\, dw$$

Which for $0 < \Re(w) < 1$, we have the formula:

$$\Gamma(w) B(1-w) = \int_0^\infty \phi(x) x^{w-1}\, dx$$

Which is expandable as:

$$\Gamma(w) B(1-w) = \sum_{n=0}^{\infty} B(n+1) \frac{(-1)^n}{n!(w+n)} + \int_1^\infty \phi(x) x^{w-1}\, dx$$

Which is an analytic continuation for $\Re(w) < 1$. Note that $B(1)$ is a convergent value because it looks like the sum $\sum_{k=1}^\infty d^k$, and we've conjugated so that $0 < d < 1$. We can summarize this, as predominantly a result by Euler and Ramanujan. For short, we'll call it the factorization lemma.

**Lemma 3.2** (The Factorization Lemma)**.** *Any element $F$ of the Gevrey space $\mathfrak{G}_0$ is expressible as:*

$$\Gamma(1-w)F(w) = \sum_{n=0}^{\infty} F(n+1)\frac{(-1)^n}{n!(1-w+n)} + \int_1^{\infty} \phi(x)x^{-w}\,dx$$

*For $\Re(w) > 0$ and the function $\phi$ is written as:*

$$\begin{aligned}\phi(x) &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(w)F(1-w)x^{-w}\,dw \\ &= \sum_{n=0}^{\infty} F(n+1)\frac{(-x)^n}{n!}\end{aligned}$$

This lemma serves to tell us, we only need a countable set of values to determine $F$ for the whole half-plane. The author has written of this frequently, and is of concurrent literature–though it is typically found in Analytic Number Theory. We, again, reference Don Zagier [8] for a break down of similar results. One would usually see $F$ as some kind of $\zeta$-function.

When we convert back into talking about $f$'s behaviour on an attracting petal $\mathcal{P}$–we have, to begin, shrunk $\mathcal{P}$ to be as close as possible to 0 (We've set $K$ arbitrarily large), we've chosen a sequence of points $\{z_k\}_{k=0}^{\infty}$ accumulating at 0, and from these points we've reconstructed $f$ on this "Neighborhood-Petal". And all of this is done with a Mellin transform.

# 4 Euler, and Borel's advanced adaptation

Diving into the relevance of the Mellin transform in producing something novel is difficult. We start with Helms' first inspiration for this topic. That is Euler's series:

$$m(u) = \sum_{k=0}^{\infty}(-1)^k k! u^k$$

Whereby:

$$m(u) = \int_0^{\infty} \frac{e^{-t}}{1+ut}\,dt$$

This becomes the basis of the borel sum–$\mathcal{B}$.

$$\mathcal{B}m(u) = \sum_{k=1}^{\infty} (-1)^k u^k = M(u)$$

Euler's series is perfect to describe the general behaviour; and can be a guiding example of what we will do. To begin though, we should avoid the end result that Euler gave. The Borel sum of $m(u)$ requires the intermediary step, of removing factorials; by which:

$$M(u) = \sum_{k=0}^{\infty} (-1)^k u^k = \frac{1}{1+u}$$

Euler got very lucky in this continuation; because there already exists an analytic continuation of this object. Without this analytic continuation–there can be no obvious form of this Borel sum. To begin the analysis, we concern ourselves with $u = 1/w$, so that:

$$m(w) = \sum_{k=0}^{\infty} (-1)^k k! w^{-k} = \int_0^{\infty} \frac{we^{-t}}{w+t}\, dt$$

This now takes the half plane $\Re(w) > 0$ into itself. And we can accurately think of $m \in \mathfrak{G}_0$. This additionally means our factorization lemma is valid. By The Factorization Lemma 3.2:

$$\Gamma(1-w)m(w) = \sum_{n=0}^{\infty} m(n+1) \frac{(-1)^n}{n!(w-1+n)} + \int_1^{\infty} \phi(x) x^{-w}\, dx$$

Where:

$$\phi(x) = \sum_{n=0}^{\infty} m(n+1) \frac{(-x)^n}{n!}$$

The difference we see from this formula, is that it analytically continues any Borel sum. Switch $m$ for $M$, and this avoids the trouble. To explain, we consider Euler's case, where:

$$\frac{w\Gamma(1-w)}{1+w} = \sum_{n=0}^{\infty} \frac{n+1}{n+2} \frac{(-1)^n}{n!(w-1+n)} + \int_1^{\infty} \phi(x) x^{-w}\, dx$$

Additionally, $\phi(x)$ is a well known function; it is simply:

$$\phi(x) = \sum_{n=0}^{\infty} \frac{n+1}{n+2} \frac{(-x)^n}{n!}$$

If we were to not know that $\frac{w}{1+w}$ was analytically continuable to $\Re(w) > 0$, we do now. This result is fairly intuitive, and seems like a carousel ride. It's just a small trick really, if you pay close attention.

We will summarize this transformation primarily as Euler wrote it, before we generalize the object. For this, we will make a restriction on our Gevrey space within this theorem.

This will be our first big theorem; which I personally do not feel as my own. I feel it is a librarian feels a theorem. I have simply combined three well known results; where I'm intended to reference results in the aether. Or, this theorem is from the literature, as compared to of my own journal.

**Theorem 4.1** (Mellin-Euler-Ramanujan Representation Theorem)**.** *Let $F(w)$ belong to the Gevrey space $\mathfrak{G}_0$, and additionally, let's ask that $F(w) = o(1)$. By which, $F$ is now written as:*

$$F(w) = \sum_{k=1}^{\infty} F_k w^{-k}$$

*Defining $G(w)$ as the function:*

$$G(w) = \sum_{k=1}^{\infty} F_k \frac{w^{-k}}{k!}$$

*Then the function $G$ is represented as:*

$$\Gamma(1-w)G(w) = \sum_{n=0}^{\infty} G(n+1)\frac{(-1)^n}{n!(1-w+n)} + \int_1^{\infty} \phi(x)x^{-w}\,dx$$

*For,*

$$\phi(x) = \sum_{n=0}^{\infty} G(n+1)\frac{(-x)^n}{n!}$$

**Remark 4.2.** It is imperative that the reader understand the transformation from Petal to half plane; and the role $K$ plays. This object, $G$, must allow for a large enough $K$, in the variable change $w = \phi_K(z)$, for the formula to be valid. We have set $K$ large enough, and bundled it into the result, solely because the end result always looks like this. This is not that strict a requirement, as all we ask is that $F_k/k! = O(d^k)$ for $0 < d < 1$–which is always possible if we choose $K$ large enough (localize the petal enough). Which through conjugation, means we can just go ahead and talk about *a* Petal, and in essence ignore $K$.

*Proof.* We start belaying a portion of this proof to a Lemma to come after this theorem. If $F$ is in the Gevrey space $\mathfrak{G}_0$, we will say that $G(w)$ is holomorphic for $\Re(w) > K'$–but additionally is holomorphic at $w = \infty$ on the Riemann sphere. The Lemma to come, is that $K' = 0$ is an appropriate choice.

Make the change of variables $G(w + K') - K' = \mathcal{G}(w)$, so that $\mathcal{G}$ is holomorphic for $\Re(w) > 0$. Which is always possible. We've done this earlier in the paper, we are just reiterating now–we can always conjugate to be about zero. The rest of the theorem speaks for itself. By The Factorization Lemma 3.2, $\mathcal{G}$ can be represented. The function $\mathcal{G}(w) = O(1)$ for $\Re(w) > 0$, and all paths to infinity.

To finalize this result, we recall if $F \in \mathfrak{G}_0$, then the growth of its coefficients are precisely $O(d^k k!)$ (which was mentioned above). And therefore the coefficients of $\mathcal{G}$ are $O(d^k)$. Which means they behave similarly as Euler's example.

By which, if $\mathcal{G}(w) = \sum_{k=0}^{\infty} F_k \frac{w^{-k}}{k!} \sim \sum_{k=0}^{\infty} O(d^k) w^{-k}$. By such, all we need are the coefficients $F_k$ to derive that:

$$\Gamma(1-w)\mathcal{G}(w) = \sum_{n=0}^{\infty} \mathcal{G}(n+1)\frac{(-1)^n}{n!(1-w+n)} + \int_1^{\infty} \phi(x)x^{-w}\,dx$$

Where:

$$\phi(x) = \sum_{n=0}^{\infty} \mathcal{G}(n+1)\frac{(-x)^n}{n!}$$

If we can set $K' = 0$ it means our original choice of $K$ for $F$, carries over into $G$–and there is no need for $\mathcal{G}$. Or rather $K' = 0$ and $\mathcal{G} = G$. (Again, this is delayed to the next section.) $\square$

We can now speak freely of Euler's Theorem; or of his representation of Borel's sums. We do nothing more than write:

$$F(w) = \int_0^{\infty} G(w/y)e^{-y}\,dy$$

By Lesbegue's interchange of integrals, and a clever identity by Euler:

$$\int_0^{\infty} y^k e^{-y}\,dy = k!$$

We observe the result as nothing but an interchange of integral and summations:

$$
\begin{aligned}
F(w) &= \sum_{k=0}^{\infty} F_k w^{-k} \\
&= \sum_{k=0}^{\infty} F_k \frac{w^{-k}}{k!} \int_0^\infty y^k e^{-y}\, dy \\
&= \int_0^\infty e^{-y} \sum_{k=0}^{\infty} \frac{F_k}{k!} \left(\frac{y}{w}\right)^k dy \\
&= \int_0^\infty G(w/y) e^{-y}\, dy \\
&= \int_0^\infty G(1/p) e^{-wp}\, dp
\end{aligned}
$$

Given the analytically continued expansion of $G$, this object always converges. And we have successfully created a one-to-one correspondence between $F \in \mathfrak{G}_0$ and $G$ holomorphic on the right half plane. The reader is to recall that we have analytically continued the expression for $F$ above. We've also artfully introduced the Laplace transform into our discussion–though this is nothing new to Borel Sums.

We will close this section with the true expansion:

$$
F(w) = \sum_{n=0}^{\infty} \frac{(-1)^n G(n+1)}{n!} \int_0^\infty \frac{1}{\Gamma(1-\frac{1}{p})} \frac{e^{-wp}}{1-\frac{1}{p}+n}\, dp + \int_0^\infty \int_1^\infty \frac{\phi(x) x^{-\frac{1}{p}} e^{-wp}}{\Gamma(1-\frac{1}{p})}\, dx dp \tag{4.1}
$$

We have not shown that this is a convergent expression as of yet, though it is built from convergent expressions. Analysing these integrals gives us the first glimpse of how the Mellin transform can be massaged to apply when talking about Borel sums. We will not muddy the pool yet, by speaking too rashly. As for that, we justify the convergence of the above integrals in the second to next section.

## 5 Mellin's proof of Euler's result

The following result is not a result by Euler or Mellin. But it is a result which expresses exactly how Mellin and Euler manipulated their results. This section will prove a lemma that will be a keystone in most of our work. We will show that $K' = 0$ and $\mathcal{G} = G$.

We write:

$$
G(w) = \sum_{k=1}^{\infty} F_k \frac{w^{-k}}{k!}
$$

Where:

$$F(w) = \sum_{k=1}^{\infty} F_k w^{-k}$$

The function $F$ is analytic for $\mathbb{C}/(-\infty,-1)$, per our construction. The thing to prove is that $G$ is analytically continuable to $\Re(w) > 0$. This is not explicitly that hard, but it is technical. And for this we have to stretch our use of Mellin transforms.

$$\begin{aligned}\widehat{F}(\xi) &= \int_0^\infty F(w) w^{\xi-1}\,dw\\ &= \sum_{k=1}^{\infty} \frac{F_k}{(\xi-k)} + \int_0^1 F(w) w^{\xi-1}\,dw\end{aligned}$$

The integral converges for $0 < \Re\xi < 1$, because $F(w) = O(1/w)$ as $w \to \infty$ and $F(w) = O(1)$ as $w \to 0$–but is analytically continuable to $\Re\xi > 0$. This means, by the basis of the Mellin transform, that:

$$F(w) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \widehat{F}(\xi) w^{-\xi}\,d\xi$$

We can express $G(w)$, using only the expression $\widehat{F}(\xi)$. We write this exactly as:

$$G(w) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\widehat{F}(\xi) w^{-\xi}}{\Gamma(1+\xi)}\,d\xi$$

This expression converges for $w \in (0,\infty)$. Additionally, since $\widehat{F}(\xi)$ has decay like $O(e^{-\pi|\Im(\xi)|})$–this is enough to show that $G(w)$ is holomorphic for $\Re(w) > 0$. We will finalize this result in the needed lemma:

**Lemma 5.1.** *Let $F(w)$ be holomorphic on $\mathbb{C}/(-\infty,-1)$, and have an expansion:*

$$F(w) \sim \sum_{k=1}^{\infty} F_k w^{-k} \text{ as } |w| \to \infty$$

*Where $|F_k| = O(d^k k!)$ for $0 < d < 1$. Then, the function:*

$$G(w) = \sum_{k=1}^{\infty} F_k \frac{w^{-k}}{k!}$$

*Is analytically continuable to $\Re(w) > 0$.*

*Proof.* We begin by writing the Mellin transform:

$$\widehat{F}(\xi) = \int_0^\infty F(w) w^{\xi-1} \, dw$$

This object is also represented by:

$$\widehat{F}(\xi) = e^{i\theta\xi} \int_0^\infty F(e^{i\theta} w) w^{\xi-1} \, dw$$

For all $-\pi < \theta < \pi$. This is nothing but a change of variables, or a use of Cauchy's theorem. Taking absolute values and letting $\Im(\xi) \to \infty$; we get:

$$\left|\widehat{F}(\xi)\right| \leq e^{-\theta\Im(\xi)} \int_0^\infty |F(e^{i\theta} w)| |w|^{\Re\xi - 1} \, dw$$

The integral converges to a bounded value, so that $\widehat{F}(\xi) = O(e^{-\pi|\Im(\xi)|})$. Therefore, from Remmert [5]:

$$\frac{\widehat{F}(\xi)}{\Gamma(1+\xi)} = O(e^{-\frac{\pi}{2}|\Im(\xi)|})$$

The function $\widehat{F}(\xi)$ is meromorphic for $\Re\xi > 0$ with simple poles at $\xi = k$ (Don Zagier [8] gives a detailed expansion on how to derive this), where the singular part at $\xi = k$ is given as:

$$\widehat{F}(\xi) = \frac{F_k}{\xi - k} + O(1)$$

Thereby, a contour integral $\tau$ about $\xi = k$, by Cauchy's integral formula, satisfies:

$$\frac{1}{2\pi i} \int_\tau \frac{\widehat{F}(\xi)}{\Gamma(1+\xi)} w^{-\xi} \, d\xi = F_k \frac{w^{-k}}{k!}$$

We can define the integral as the sum of the residues (The contour integral converges in the right half plane by $\Gamma$ and $\widehat{F}$'s asymptotics), whereby:

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\widehat{F}(\xi)}{\Gamma(1+\xi)} w^{-\xi} \, d\xi = \sum_k \mathrm{Res}_{\xi=k} \frac{\widehat{F}(\xi)}{\Gamma(1+\xi)} w^{-\xi} = \sum_{k=1}^\infty F_k \frac{w^{-k}}{k!} = G(w)$$

Finally, to show that $G$ is holomorphic for $\Re(w) > 0$, we need to only identify that the integral on the left of the above equation converges for $\Re(w) > 0$. This is because of the decay we showed earlier in this proof. The function $w^{-\xi}$ grows at worst like $O(e^{\kappa|\Im(\xi)|})$ for $\kappa < \pi/2$, and $1/\Gamma(1+\xi) = O(e^{\frac{\pi}{2}|\Im(\xi)|})$ for $\Re(\xi) = \frac{1}{2}$ (See Sterling's Asymptotic Theorem in the complex plane–from Reinhold Remmert's Classical Topics in Complex Function Theory [5]). While additionally, $\widehat{F}(\xi) = O(e^{-\pi|\Im(\xi)|})$. Then, what's left over is a decay like $e^{(\kappa - \frac{\pi}{2})|\Im(\xi)|}$–which surely converges under the infinite integral. ☐

So, to say this is Mellin's proof of Euler's result, is to say that. If Euler didn't know that:

$$\frac{w}{1+w}$$

was holomorphic for $\Re(w) > 0$–to perform his famous divergent sum. He would now. This is the point of this theorem, to analytically continue something that may not be obvious.

## 6 Understanding the integrals in Equation 4.1

**The first integral**

The first thing that needs to be identified, that may not have been apparent–is that there are no singularities in $1/\Gamma(1-1/p)$ except at $p = 0$. And that there are zeroes precisely at $p = 1/k$ for $k \in \mathbb{N}$–$k \geq 1$. This means that:

$$\frac{1}{\Gamma(1-\frac{1}{p})} \frac{e^{-wp}}{1-\frac{1}{p}+n} \neq \infty \text{ When } \Re(w) > 0,\ n \geq 0,\ p \in (0, \infty)$$

This is because the simple pole at $1 - 1/p_n + n = 0$ is cancelled by the simple zero of $1/\Gamma(1 - 1/p_n) = 0$. Where $p_n = 1/(1 + n)$. This means, we only have to ask if:

$$\psi_n(p) = \frac{1}{\Gamma(1-\frac{1}{p})(1-\frac{1}{p}+n)}$$

Has a convergent Laplace transform. At $p \to \infty$, we are absolutely fine, as $\psi_n = O(1)$. But as $p \to 0$, we encounter more difficulty. The object has a clear singularity at $p = 0$. What our saving grace is, is that the singularity is an essential singularity, and thereby has paths of convergence. What we hope for, is that $(1 - 1/p + n)$'s zero at $p = 0$ over powers $1/\Gamma(1 - 1/p)$'s singularity–if we choose an appropriate path.

The short form it doesn't converge when $p \in \mathbb{R}^+$. But does converge if $p \in \gamma$, where $\gamma$ is a specific kind of curve. That curve is given as a curve which starts at zero, and leaves tangentially in the upper left half plane, from which it travels straight to $+\infty$. Without introducing more advanced techniques from Laplace transforms, and Mellin transforms–we want to keep things relatively simple, but this is still a valid Laplace transform.

$$||\psi_n(p)||_{p\in\gamma} < \infty$$

And this can be bounded uniformly $M \geq \sup_{n\in\mathbb{N}} ||\psi_n(p)||_{p\in\gamma}$. And we can write, exactly what we mean for the first integral:

$$\int_0^\infty \psi_n(p)e^{-wp}\,dp = \int_\gamma \psi_n(p)e^{-wp}\,dp$$

Where the integral in the first object does not converge, but does so in the second object. This appears very frequently in Laplace transforms–we change the path of the integral; this path is perfectly fine, and we have nothing to worry about. All our work above can be substituted for this curve. After this; we can identify that:

$$\Psi_n(w) = \int_\gamma \psi_n(p) e^{-wp}\, dp$$

And that:

$$|\Psi_n(w)| < M \int_0^\infty e^{-Re(\gamma(t)w)} |\gamma'(t)|\, dt$$

This implies that $\Psi_n = O(1)$. Thereby:

$$\sum_{n=0}^\infty \frac{(-1)^n G(n+1)}{n!} \Psi_n(w) = \sum_{n=0}^\infty \frac{(-1)^n G(n+1)}{n!} O(1)$$

Which proves that this function converges uniformly. By which we mean for all $\Re(w) > 0$; there is a compact neighborhood where the sum:

$$\sum_{n=0}^\infty \frac{(-1)^n G(n+1)}{n!} \Psi_n(w)$$

Is a valid function.

### The second integral

It's a little trickier to get the second integral, we start with the same identification, that instead of integrating over $p \in (0, \infty)$ we choose $p \in \gamma$. Then the integral looks like:

$$\int_\gamma \int_1^\infty \frac{\phi(x) x^{-\frac{1}{p}} e^{-wp}}{\Gamma(1 - \frac{1}{p})}\, dxdp$$

We note firstly that:

$$\int_1^\infty \phi(x) x^{-\frac{1}{p}}\, dx = U(p)$$

And $U(p)$ is holomorphic for $\Re(1/p) > -\delta$. The function $\phi(x)$, can again be modified by choosing large enough $K$, which is again "shrinking the Petal to be as local as possible". This gives us a bound of the form $\phi(x) = O(x^{-1-\delta})$ where $\delta = \delta(K)$.

Now there is a small notch in $\gamma$, where it touches the upper left half plane, call the small negative real part value it achieves $-\epsilon$. The value $\epsilon$ and $K$ can be chosen so that $\delta > \epsilon$.

Thereby, we have that:

$$\int_\gamma \frac{U(p)e^{-wp}}{\Gamma(1-\frac{1}{p})}\,dp$$

Is a convergent integral, as $U(p)$ is holomorphic on the arc $\gamma$–while additionally being bounded by at worst $O(p)$ on this curve.

This gives us our first strong representation of our function $F$–which performs a Borel sum.

**Theorem 6.1** (The Mellin-Borel Sum). *Let $F$ belong to the Gevrey space $\mathfrak{G}_0$, and assume that $F$ has the expansion:*

$$F(w) = \sum_{k=1}^\infty F_k w^{-k}$$

*Define the functions:*

$$G(w) = \sum_{k=1}^\infty F_k \frac{w^{-k}}{k!}$$

*And:*

$$\phi(x) = \sum_{n=0}^\infty G(n+1)\frac{(-x)^n}{n!}$$

*Let $\gamma$ be a curve from $0$ to $+\infty$, with a small notch into the upper left half plane from $0$. Then:*

$$F(w) = \sum_{n=0}^\infty \frac{(-1)^n G(n+1)}{n!}\int_\gamma \frac{1}{\Gamma(1-\frac{1}{p})}\frac{e^{-wp}}{1-\frac{1}{p}+n}\,dp + \int_\gamma\int_1^\infty \frac{\phi(x)x^{-\frac{1}{p}}e^{-wp}}{\Gamma(1-\frac{1}{p})}\,dxdp$$

*Which converges for $\Re(w) > 0$.*

# 7 Converting back through the change of variables

As a good amount of our change of variables were implicit, we walk the reader through the various change of variables we have have performed. We began with the function: $f(z)$ which is entire, with a fixed point at 0. We have assumed the value $f'(0) = 1$. We then wrote $f$ in the expansion:

$$f(z) = z + az^{\ell+1} + O(z^{\ell+2}) = z\left(1 + az^\ell + O(z^{\ell+1})\right)$$

The value $\ell + 1$ was called the *multiplicity* by Milnor; and the value $\ell$ was called the *valit* by Trappmann. We then defined $w = \varphi(z) = \frac{-1}{\ell a z^\ell}$, and $z = \varphi^{-1}(w) = \frac{1}{\sqrt[\ell]{-\ell a w}}$. We have then made the change of variables:

$$F(w) = \varphi \circ f \circ \varphi^{-1}(w)$$

This function takes $\Re(w) > K$ to itself; whereby, we have chosen $K$ large enough, and made the change of variables:

$$\mathcal{F}(w) = F(w + K) - K$$

We chose to write this simply as $F$, as we allowed ourselves to ignore the value $K$, this isn't too hard to envision, if we roll this into $\varphi$. This becomes a perhaps slightly clearer change of variables. Which we'll write as: $w = \varphi_K(z) = \frac{-1}{\ell a z^\ell} - K$, and $z = \varphi_K^{-1}(w) = \frac{1}{\sqrt[\ell]{-\ell a(w+K)}}$.

Whereby, the actual change of variables we performed was:

$$F(w) = \varphi_K \circ f \circ \varphi_K^{-1}(w)$$

This is a degree more convenient than the original, Ecalle, change of variables. Though, this is only true for applications to Mellin transforms–and absolutely not in general. In a sense, it's the Ecalle change of variables for purposes of Mellin.

We can now see quite clearly how The Mellin-Borel Sum 6.1 can uncover the actual value of $f$. Although, it will only be holomorphic in $z$ in a small neighborhood petal about 0.

# 8 Applications for the general sense

The entire purpose of this exchange, and this construction, was to jusify Euler's representation of Borel's sums–but now applied to fractional iterations. This also extends to the Abel function itself, though that is a tad more difficult. We'll begin by taking fractional iterations. For that, we will argue by example with the half-iterate, though this works for any fractional iterate.

Let $h(h(z)) = f(z)$, and let us make the change of variables:

$$H(w) = \varphi_K \circ h \circ \varphi_K^{-1}(w)$$

We know that:

$$H(w) = w + \frac{1}{2} + \sum_{k=1}^{\infty} H_k w^{-k}$$

Where $H_k = O(d^k k!)$, where $0 < d < 1$. Which we can write as:

$$H(w) = P(w) + o(1)$$

Which means it's in the Gevrey space. We can ignore the term $P(w)$, as this is easily calculable–instead, solely focusing on the $o(1)$ term, which we'll give the nomer $B(w)$. Then defining the two functions:

$$G(w) = \sum_{k=1}^\infty H_k \frac{w^{-k}}{k!}$$

$$\phi(x) = \sum_{n=0}^\infty G(n+1)\frac{(-x)^n}{n!}$$

Then, the initial transformation is that:

$$\Gamma(1-w)G(w) = \sum_{n=0}^\infty G(n+1)\frac{(-1)^n}{n!(n+1-w)} + \int_1^\infty \phi(x)x^{-w}\,dx$$

Which is holomorphic for $\Re(w) \geq 0$, and which allows us to take the Borel sum unscrupulously:

$$B(w) = \sum_{n=0}^\infty \frac{(-1)^n G(n+1)}{n!}\int_\gamma \frac{1}{\Gamma(1-\frac{1}{p})}\frac{e^{-wp}}{1-\frac{1}{p}+n}\,dp + \int_\gamma\int_1^\infty \frac{\phi(x)x^{-\frac{1}{p}}e^{-wp}}{\Gamma(1-\frac{1}{p})}\,dxdp$$

Whereby, we write $H(w) = w + \frac{1}{2} + B(w)$. And now, undoing the Ecalle-Mellin change of variables; we have an analytic expression for $h(z)$ in a neighbhorhood petal of 0–given all we've used are the coefficients $H_k$, which were derived from $h_k$. We can find these coefficients very rapidly using Helms-Trappman Matrix Algorithm.

For the Abel function $\alpha(f(z)) = \alpha(z) + 1$, things are a tad trickier. We note that:

$$A(w) = w + C\log w + \sum_{k=0}^\infty A_k w^{-k}$$

For $A(F(w)) = A(w) + 1$. This is solvable using a similar coefficient grab as before. Which is something Helms would write using a matrix equation; as we are simply solving a system of linear equations. And thanks to work by greater mathematicians than myself, we have $A_k = O(d^k k!)$ for $0 < d < 1$. Which leaves us in the same territory to apply The Mellin-Borel Sum 6.1.

# 9 Applying the Mellin transform in an alternative manner

There exists an alternative manner of expressing the half iterate $h$, which highlights more of the malleability of this method, in comparison to the more standard approach. As we have detailed, through the portions of this paper, we can now identify that:

$$f^{\circ s}(z) : \mathbb{C}_{\Re(s)>0} \times \mathcal{P} \to \mathcal{P}$$

For an appropriately chosen petal $\mathcal{P}$. Where additionally, $f^{\circ s} = O(s)$. This allows us to apply a similar form of the Factorization Lemma 3.2; though it is a tad stranger.

Let:

$$\vartheta(x,z) = \sum_{n=0}^{\infty} f^{\circ n+1}(z) \frac{(-x)^n}{n!}$$

Then:

$$\Gamma(1-s) f^{\circ s}(z) = \sum_{n=0}^{\infty} f^{\circ n+1}(z) \frac{(-1)^n}{n!(n+1-s)} + \int_1^{\infty} \vartheta(x,z) x^{-s}\, dx$$

This is again, nothing more than a result by Euler and Ramanujan. Any function which is bounded $Q(s) = O(e^{\rho|\Re(s)|+\tau|\Im(s)|})$ for $\tau, \rho > 0$ while $\tau < \pi/2$ is representable in this manner (given $Q$ is holomorphic in the right half plane). Thereby, just as well is $f^{\circ s}$ bounded in such a manner.

So, for example, if we take $h(z)$, then:

$$h(z) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\vartheta(x,z)}{\sqrt{x}}\, dx$$

This expansion is valid for $z$ in any attracting petal about 0. It is important to note now, as we haven't quite yet. That to rigorously derive the above expression, we needed to take the detour through Borel sums; as this allowed us to construct the function:

$$f^{\circ s}(z) = \alpha^{-1}\left(\alpha(z) + s\right)$$

in a rigorous setting. This then further let us get the estimate $O(s)$–which is absolutely pivotal to this expansion. This capitulates most of the work we have done here. Everything is obsolesced by the quick convergence of the above expression. The author has always known this expression to exist, but could never justify its existence.

## 10 Numerical evidence, and Strong heuristics

Let us start this discussion with a primal use of our integral expression. We call this a strong analytic proof:

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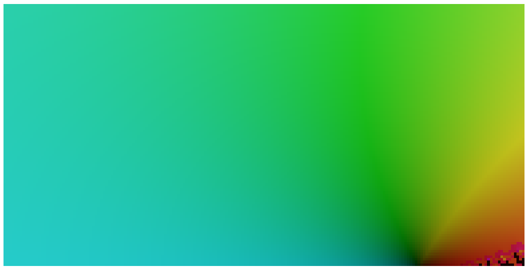

Figure 1: A graph of $h(z)$ where $h(h(z)) = e^z - 1$, done over $-2 < \Re(z) < 0.5$, and $0 < \Im(z) < 1.25$.

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