Introduction to the non-integer ranks problem*

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We give here a brief introduction to the **non-integer ranks problem** for the Goodstein's Hyperoperations defining it and explaining how it has been attacked in the last decades. What follows is adapted from a Math Stack Exchange answer [0] that aims to be a collection of references and a place where one can start thinking about the subject of hyperoperations.

The question

> [If] $\uparrow^n$ and $G(n,\cdot,\cdot)$ are notations for hyperoperation. [...] n is the hyperoperations rank. Can example x, y and z values be provided for either the following formulae [...]
>
> $$z = x\uparrow^{-3} y \qquad \text{or} \qquad z = G(-1,x,y)?$$
>
> Or, alternatively, for the formulae
>
> $$z = x\uparrow^{-0.5} y \qquad \text{or} \qquad z = G(1.5,x,y)?$$

When we ask if values of $G(n,-,-)$ can be provided if $n \in \mathbb{Z}$ or $n \in \mathbb{R}$ we are actually asking if is possible to define an extension of G to the integers, rational or real numbers (or complex) satisfying the Hyperoperations recursion over all the domain.

Brief introduction

Notation 1: With $G : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{N}$ we mean the Goodstein function [1] or equivalently the Hyperoperations sequence: *let's work with G (a 3-ary function) as if it is an indexed family of binary functions $+_{s\in\mathbb{N}} : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$* and call the index s the rank of the hyperoperation.

$$m +_s n := G(s,m,n)$$

Notation 2: The infix notation that I'm using ($+_s$) is not common at all, as the Goodstein's G. Usually the most common notations for $+_s$ are H_s (Wikipedia page prefix notation) or $[s]$ (Square bracket infix notation) or maybe $A_s(m) = 2[s]m$ (comes from the Ackermann function and is used in some recursion papers about the Grzegorczyk hierarchy and in some recent works of D. Kouznetsov [6]) and the Knuth's Uparrow notation $\uparrow^n$ (that has a different indexing starting from $\uparrow^0 = \times$) [2]

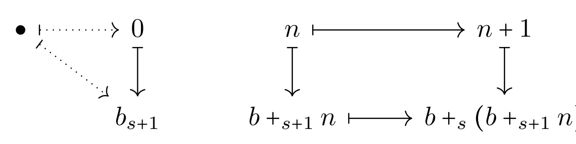
$$H_s(x,y) = x[s]y = x\uparrow^{s-2} y = G(s,x,y) = x +_s y$$

Definition 1 (Hyperoperations sequence)**.** *We define the indexed family $\{+_s\}_{s\in\mathbb{N}}$ recursively over the natural numbers ($b,n,s \in \mathbb{N}$)*

i) $$b +_0 n = n+1$$

ii) $$b +_{s+1} 0 = b_{s+1}$$

iii) $$b +_{s+1} (n+1) = b +_s (b +_{s+1} n)$$



Where base values b_{s+1} that give us the "natural/classical" hyperoperations sequence are the fol-

*This is an improved version of the answer given in [0] in 2015 *lowing*

$$iv) \qquad b_{s+1} := \begin{cases} b, & \text{if } s = 0 \\ 0, & \text{if } s = 1 \\ 1, & \text{if } s > 1 \end{cases}$$

I will call the argument b **base**, s **rank** and n **exponent**.

Remark 0 Note that this definition gives us the "standard" hyperoperations:

$$H_1(x,y) = x +_1 y = x + y$$
$$H_2(x,y) = x +_2 y = xy$$
$$H_3(x,y) = x +_3 y = x^y \qquad (Exponentiation)$$
$$H_4(x,y) = x +_4 y = {}^y x \qquad (Tetration)$$

Terminology 1: About the use of the term **rank** for the argument s. I don't think it is official but at least it makes sense for various reasons. As far as I know the term was introduced by K.A. Rubtsov and G.F. Romerio in different papers/reports for the first time since the year 2006 [3, pg 3] and the term has been widely adopted by the *Tetration Forum* in the subsequent years. Another good reason is that every Hyperoperation H_n belongs to the class $\mathcal{E}^n$ of the Grzegorczyk Hierarchy [4], a *sub-recursive hierarchy* that organizes the Primitive Recursive functions according to their growth rate. Since the position inside a hierarchy is usually denoted by the term rank (See V_α hierarchy of sets for example) it seems to me a perfect choice. But for the **base** and **exponent** terms I'm not sure: Rubtsov and Romerio tried to introduce, and used systematically, the terms **hyperbase** for b and **hyperexponent** for n together with an uniform terminology for right and left inverse hyperoperations [5].

Remark 1: This sequence follows from original Goodstein definition and it implies that the 0-th rank hyperoperation trivially coincides with the successor function $a +_0 n = n + 1$ and so does all the integers-negative ranks: more about this can be found in the good David K. answer here [0], in my and Ibrahin Tencer's answers here [7] and on the Tetration Forum [8] Anyways it is possible to avoid the imposition of $a +_0 n = n + 1$ with alternative definitions of the hyperoperations sequence that give us more freedom for the negative ranks: about this there is a large amount of work by Rubtsov and Romerio under the name of Zeration [3][1], [9], [10] and by Cesco Reale in [11]. The topic is quite controversial and not very well known so I suggest you those two thread on TetrationForum [12], [13].

Back the initial question, in my opinion it is important to notice that is unlikely that one can extend the rank to non-ingers values without finding a way to extend the base and the exponent too. If we look at the sequence $x_s := 2 +_s n$, for a fixed n , we have that $x_s \in \mathbb{N}$ if the rank is a natural number but if we want it to be continuous or analytic in the variable s, thus extending to $s \in \mathbb{R}$, is very likely that for most of the non-integers ranks x_s will have non-integer values making the functions $+_s$ not closed on the naturals for most of the non-integers ranks even when the base and exponent are natural numbers.

Example: if $2 +_{1.5} 3 = q$ and $q \in \mathbb{R} \setminus \mathbb{Z}$ then evaluating $2 +_{1.5} 4 = q$ would need to know how to evaluate $2 +_{0.5} q$

The higher-order function iteration approach

An interesting way [14] to continue is to look at some suitable space of binary functions $\mathcal{H}$ with $+_s \in \mathcal{H}$ together with a function $\Sigma : \mathcal{H} \to \mathcal{H}$ with the following property

$$\Sigma[+_s] = +_{s+1}$$

[1] www.rotarysaluzzo.it/Z_Vecchio_Sito/filePDF/Iperoperazioni%20(1).pdf

and continue investigating its dynamics because it turns out that the Hyperoperations are the natural iterations of this map Σ applied to the 0-th rank hyperoperation

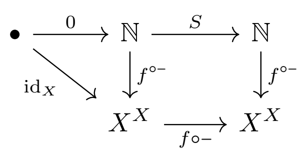
$$\Sigma^{\circ n}[+_0] = +_n$$

The operator Σ increases by one the rank of the hyperoperations so it's plausible to expect that the fractional/real/complex iteration of this map Is going to give us the fractional/real/complex rank hyperoperations:

$$\forall \sigma \in \mathbb{C}(\Sigma^{\circ\sigma}[+_0] = +_\sigma)$$

Terminology 2: The iteration we are talking about $-^{\circ n}$ can be defined recursively in the following way: given a function $f : X \to X$

$$i) \qquad f^{\circ 0}(\beta) = \beta \quad or \quad f^{\circ 0} = \mathrm{id}_X$$
$$ii) \quad f^{\circ n+1}(\beta) = f(f^{\circ n}(\beta)) \quad or \quad f^{\circ n+1} = f \circ f^{\circ n+1}$$



My naive opinion is that if one is able to find a good space $\mathcal{H}$ such that Σ is an operator then we could try to apply the powerful tools of operator theory.

This kind of point of view can be used for a larger class operation sequences. The **idea of reducing the non-integer ranks problem to a non-integer iteration problem** is, again, as far as I know, due to H. Trappmann [15](2008), the founder of the Tetration Forum . Some years later this idea was better developed by J. Nixon (2011) with the concept of "meta-superfunctions" [16] who is still working on this point of view (see later).

To explain better this we have to first find what kind of map is or should be Σ, the map that increases the rank by one unit: the discourse is a bit long so I'll send you to one of my answer at MSE [17]. As you can read in my answer, Σ is closely related to the process of iterating a function, also called "taking the **superfunction** [24]" in the case of 1-ary functions and is also closely related to known problems as finding the solutions of **Abel functional equations** [25] of the form $\chi(z) + 1 = \chi(f(z))$, **Schröder's equations** [26] $s \cdot \Psi(z) = \Psi(f(z))$.

A massive amount of research on finding/building unique superfunctions [20] was made and is still carried by D. Kouznetsov [18], [19] and you can find most of his work in this online encyclopedia [23].

Finding an unique β-based superfunction actually give us an "higher order" (because send functions to functions) function that maps the function $f(x)$ to the function $F(z) = f^{\circ z}(\beta)$

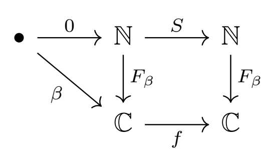
Definition 2. *Given a function f we define intuitively its β-based superfunction F_β as the function that maps to every z the z-th application of f to β*

$$F(z) = f^{\circ z}(\beta)$$

Proposition 1. *The β-based superfunction of f satisfies this equations*[2]*:*

i) $F_\beta(0) = \beta$

ii) $F_\beta(z+1) = f(F_\beta(z))$



Definition 3. *Given a suitable collection functions $\mathcal{H}$ we define intuitively the β-based superfunction map as a function $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ that maps to every f its β-based superfunction F_β, i.e. $\mathcal{S}_\beta : f \mapsto F_\beta$*

$$\mathcal{S}_\beta[F](z) = f^{\circ z}(\beta)$$

[2]Proving i) reduces to proving that $f^{\circ 0} = \mathrm{id}$ and proving 2) relies on the condition $f^{\circ w+z} = f^{\circ w} \circ f^{\circ z}$ and $f^{\circ 1} = f$. Those properties should hold for any sensible definition of non integer iteration.

Is easy to see what this has to do with Hyperoperations and with the Σ. Lets define the sequence of "**hyper-exponentiations**"!

Definition 4. *Define the family* $\{H_{b,s}\}_{b,s\in\mathbb{N}}$ *of* ***hyper-exponentiations*** *as follows*

$$H_{b,s}(n) := b +_s n$$

We have that $H_{b,1}(n) = \text{add}_b(n) = b + n$, $H_{b,2}(n) = \text{mul}_b(n) = bn$ and $H_{b,3}(n) = \exp_b(n) = b^n$.

Proposition 2. *For every* $b, s, n \in \mathbb{N}$

$$H_{b,s+1}(n+1) = H_{b,s}(H_{b,s+1}(n))$$

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{S} & \mathbb{N} \\ {\scriptstyle H_{s+1}}\downarrow & & \downarrow{\scriptstyle H_{s+1}} \\ \mathbb{N} & \xrightarrow[H_s]{} & \mathbb{N} \end{array}$$

In other words we have that the superfunction map is the map Σ we are looking for because we have that every $(s+1)$-rank **hyper-exp** is the superfunction of the s-rank **hyper-exp**, i.e. we have the

Corollary 1. $\mathcal{S}[H_{b,s}] = H_{b,s+1}$.

Conjecture 1. *At this point it makes sense to ask if finding the non-integers iterations of* $\mathcal{S}_\beta$ *really gives us non-integer ranks* ***hyper-exponentiations*** *functions*[3] *for some bases* b*, for* $\sigma \in \mathbb{R} \setminus \mathbb{N}$

$$\boxed{\mathcal{S}^{\circ\sigma}_{\beta=1}[H_{b,2}] = H_{b,2+\sigma}}$$

Remark 2: In the equation above I've setted $H_{b,2}$ and $\beta = 1$ because for all the natural ranks $s \geq 3$ we have that $H_{b,s}(0) = 1$ by definition.

Question 1. Is it possible to find the unique... real... non-integer iterations of $\mathcal{S}$?

As far as I know the answer to this question is still unknown or at least unknown to me, an amateur mathematician, even if some hot posts by the user JmsNxn appeared in the last two months on the Tetration Forum [21], [22].

[3] By a massive abuse of notation this can thought as $\mathcal{S}_{H_{2,b}}[\mathcal{S}_{\beta=1}](\sigma) = H_{b,2+\sigma}$.

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