alization of both the classical *superfunction trick* (ST) and of *Nixon's trick* (NT).

Let's begin by setting some notation and by stating a ”fake theorem” that illustrates my sentiment about this matter better than words.

1.1.2 Preliminaries on notation (and basic concepts)

From now on I will call ”a function” only a binary relation that is everywhere defined and single-valued, i.e. for every element in the domain exists a unique element in the codomain and so on. I'll denote functional composition by juxtaposition $gf = g \circ f$ and integer iteration[2] by f^n. With s I vaguely mean the successor endomap $a \mapsto a+1$ and with mul_a the endomap that scales by $x \mapsto a \cdot x$. Given a morphism $f : X \to Y$ and two morphisms in the opposite direction $i, r : Y \to X$: we call i a *section* of f if and only if $fi = \mathrm{id}_Y$, i.e. f has a right inverse; we call r a *retraction* of f if and only if $\mathrm{id}_X = rf$, i.e. f has a left inverse. As a set 1 will denote the singleton and for a generic set X the function $!_X : X \to 1$ will be the unique function from X to the singleton: the subscript in $!_X$ will be omitted when is clear what the is the intended X.

Given two endofunctions $f : X \to X$ and $g : Y \to Y$ define the set $[f, g] \subseteq Y^X$ as the solution set of the equation

$$\chi f = g\chi \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi"'] & X \arrow[d, "\chi"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

For every triple of endofunctions $f : X \to X$, $g : Y \to Y$ and $h : Z \to Z$ we can compose a solution $\psi \in [g, h]$ with a solution $\chi \in [f, g]$ and obtain a solution

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[2] The integer iteration of a function is defined by recursion or can be equivalently defined as the unique monoid morphism $f^{-} : \mathbb{N} \to X^X$ from the natural numbers under additions to the monoid of endofunction under composition s.t. $f^1 = f$.

$\psi \circ \chi \in [f, h]$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\chi"] \arrow[dd, bend right, "\psi\circ\chi"'] & & X \arrow[dr, "\chi"] \arrow[dd, bend right] & \\
& Y \arrow[rr, "g"] \arrow[dl, "\psi"] & & Y \arrow[dl, "\psi"] \\
Z \arrow[rr, "h"] & & Z &
\end{tikzcd}$$

This means that for every composable f, g, h we have a function

$$\circ : [g, h] \times [f, g] \to [f, h]$$

as a corollary we get a right monoid action of $[g, g]$ and a left monoid action of $[f, f]$ on the set of solutions $[f, g]$, i.e. let $\sigma \in [f, f]$ and $\varepsilon \in [g, g]$, if $\chi \in [f, g]$ then $\chi\sigma$ and $\varepsilon\chi$ belong to $[f, g]$.

We further define two new subsets of sequences $[f, g]_\Delta \subseteq (Y^X)^{\mathbb{N}}$ and $[f, g]_\Delta^{op} \subseteq (Y^X)^{\mathbb{N}}$ as the set of sequences $\phi_n \in (Y^X)^{\mathbb{N}}$ as follows: the first set $[f, g]_\Delta := \{\phi_n : \phi_{n+1} f = g\phi_n\}$ contains the solutions of

$$\phi_{n+1} f = g\phi_n \qquad \begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_0"] & X \arrow[r, "f"] \arrow[d, "\phi_1"] & \dots \arrow[r, "f"] & X \arrow[r, "f"] \arrow[d, "\phi_n"] & X \arrow[r, "f"] \arrow[d, "\phi_{n+1}"] & \dots \\
Y \arrow[r, "g"] & Y \arrow[r, "g"] & \dots \arrow[r, "g"] & Y \arrow[r, "g"] & Y \arrow[r, "g"] & \dots
\end{tikzcd}$$

the second set $[f, g]_\Delta^{op} := \{\phi_n : \phi_n f = g\phi_{n+1}\}$ is the dual version of the first

$$\phi_{n+1} f = g\phi_n \qquad \begin{tikzcd}
\dots \arrow[r, "f"] & X \arrow[r, "f"] \arrow[d, "\phi_{n+1}"] & X \arrow[r, "f"] \arrow[d, "\phi_n"] & \dots \arrow[r, "f"] & X \arrow[r, "f"] \arrow[d, "\phi_1"] & X \arrow[d, "\phi_0"] \\
\dots \arrow[r, "g"] & Y \arrow[r, "g"] & Y \arrow[r, "g"] & \dots \arrow[r, "g"] & Y \arrow[r, "g"] & Y
\end{tikzcd}$$

Notice that if we define $\tilde{\phi}(n, x) := \phi_n(x)$, the condition $\phi \in [f, g]_\Delta$ implies that $\tilde{\phi}$ is a solution of the equation

$$\tilde{\phi}(n+1, f(x)) = g(\tilde{\phi}(n, x))$$

and respectively $\phi \in [f,g]_{\Delta}^{op}$ implies that $\tilde{\phi}$ is a solution of the equation

$$f(\tilde{\phi}(n+1,x)) = \tilde{\phi}(n,g(x))$$

Ok! We are now ready for the a first statement of the generalized superfunction trick

1.1.3 Road to the generalized superfunction trick

GST theorem. *Given functions* $f : X \to X$, $g : Y \to Y$ *and a sequence of functions* $\phi : \mathbb{N} \to Y^X$ *if the conditions*

1. *for every natural number,* $\phi_n f = g\phi_{n+1}$ *or* $\phi_{n+1} f = g\phi_n$;
2. ϕ_0 *is "appropriate";*

are met, then the "limit" of the sequence $\phi_n \to \chi$ *exists and lands in the subset* $[f,g] \subseteq Y^X$, *i.e.*

$$\chi f = g\chi$$

The functioning of this theorem depends fundamentally on the existence and on our ability to build sequences of maps satisfying (1) and secondarily on our ability to build to do so in a way that ensure us the resulting sequence converges in some sense. Luckily, the mere existence of sequences satisfying (1) is not a problem! Such kinds of sequences exist, are definable by recursion and are abundant: in fact we can prove these

Easy Lemmas. *Given functions* $f : X \to X$ *and* $g : Y \to Y$. *For every function* $\phi : X \to Y$ *we prove that:*

1. f *is split-mono* $\Rightarrow$ *exists a sequence* α_n *s.t.* $\alpha_0 = \phi$ *and* $\alpha_{n+1} f = g\alpha_n$;
   f *is split-epi* $\Rightarrow$ *given* $\alpha_n, \alpha'_n \in [f,g]_{\Delta}$ *if* $\alpha_0 = \alpha'_0$ *then* $\alpha_n = \alpha'_n$;
   if f *is iso* $Y^X \simeq [f,g]_{\Delta}$
2. g *is split-epi* $\Rightarrow$ *exists a sequence* β_n *s.t.* $\beta_0 = \phi$ *and* $\beta_n f = g\beta_{n+1}$;
   g *is split-mono* $\Rightarrow$ *given* $\alpha_n, \alpha'_n \in [f,g]_{\Delta}^{op}$ *if* $\alpha_0 = \alpha'_0$ *then* $\alpha_n = \alpha'_n$;
   if g *is iso* $Y^X \simeq [f,g]_{\Delta}^{op}$
3. *if* $\phi \in [f,g]$ *the constant sequence* $\phi!$ *is in both* $[f,g]_{\Delta}$ *and* $[f,g]_{\Delta}^{op}$;