

# Solving Conjugacy Equations on $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$

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## Abstract

The author constructs a space of functions  $\mathcal{B} \subset \mathcal{C}^1(\mathbb{R}, \mathbb{R}) = \{f(x) \mid f : \mathbb{R} \rightarrow \mathbb{R}, f'(x) : \mathbb{R} \rightarrow \mathbb{R}\}$  which satisfies the conjugacy property. The conjugacy property can be summarized as  $\forall f, g \in \mathcal{B} \ g > f \ \exists \phi \in \overline{\mathcal{B}}$  s.t.  $g(\phi(x)) = \phi(f(x))$ . This paper is intended as an exercise in functional analysis, and used as an existence method of solutions to function spaces with the conjugacy property.

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## 1 Introduction

This paper will be a brief notice, and not much more. We intend to construct a manner of designing conjugacy classes under the composition operator. For this reason, we're going to play with toy-models; very simple cases which should help us with more complicated functions. Much of this paper originates from a series of discussions the author had with MphLee on the Tetration forum. This is intended as a proof of concept of much of what we discussed.

For this reason, we're going to work in the space  $\mathcal{C}^1(\mathbb{R}, \mathbb{R}) = \{f(x) \mid f : \mathbb{R} \rightarrow \mathbb{R}, f'(x) : \mathbb{R} \rightarrow \mathbb{R}\}$ , and pull out a subset of this space which satisfies the conjugacy property. The first thing we'll want is that,

$$f'(x) > 0$$

So that there is an inverse function  $f^{-1}$  which is also differentiable. And, to pair hand in hand with this, we also want that,

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ bijectively}$$

Which, can be summarized as  $f : \mathbb{R} \rightarrow \mathbb{R}$  is bijective with positive derivative. The next things we need from  $f$  to construct the space  $\mathcal{B}$ , we want some slightly different requirements. The first thing we want is that,

$$f'(x) \rightarrow A > 1 \text{ as } x \rightarrow \infty$$

Where  $A \in \mathbb{R} \cup \infty$ . So that the derivative grows past 1. This being very important in the manner we construct our conjugate functions; but is less necessary than the above conditions. But still, in the toy model, saves us a bunch of space. The last thing we need, is even more artificially constructed. It is the idea that,

$$f^{\circ-n}(x) \rightarrow -\infty$$

For all  $x \in (-\infty, \infty)$ . This is that the orbits of  $f^{-1}(x)$  tend to negative infinity. We need this condition to ensure that  $\phi \in \mathcal{B}$ . If we were to drop this condition, we could still construct  $f(\phi(x)) = \phi(g(x))$  but it is no longer necessary that  $\phi \in \mathcal{B}$ .

We will call this our set  $\mathcal{B}$ . Which is given as,

$$\mathcal{B} = \{f(x) \mid f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}), f'(x) > 0, f \text{ is a bijection, } f'(x) \rightarrow A > 1 \text{ as } x \rightarrow \infty, f^{\circ-n}(x) \rightarrow -\infty \text{ as } n \rightarrow \infty\}$$

This set  $\mathcal{B}$  is a monoid under composition. This is to mean, for  $f, g \in \mathcal{B}$  we have  $f(g) \in \mathcal{B}$ . Which the only thing that is non-trivial is that the orbits tend to negative infinity, but it's apparent because  $f^{-1}(x) < x$  eventually, and so must  $g^{-1}(f^{-1}(x)) < x$ , and so the result happens similarly.

Therein, the main result of this notice, is to show that  $\mathcal{B}$  satisfies the conjugacy property. Which is,

$$\forall f, g \in \mathcal{B} : \exists \phi \in \overline{\mathcal{B}} f(\phi(x)) = \phi(g(x)) \text{ or } \exists \phi \in \overline{\mathcal{B}} g(\phi(x)) = \phi(f(x))$$

Where  $\overline{\mathcal{B}}$  is the closure of  $\mathcal{B}$  to include  $\lim_{x \rightarrow \infty} f'(x) = 1$ . The manner we do this is in a chain of constructions. We begin by constructing,  $\Phi(x) : \mathbb{R} \rightarrow \mathbb{R}^+$ ; such that  $\Phi(x+1) = e^x f(\Phi(x))$ . The second construction is to use  $\Phi$  to construct  $F : \mathbb{R} \rightarrow \mathbb{R}$  such that  $F(x+1) = f(F(x))$ . And then, constructing in the same manner a function  $G$  such that  $G(x+1) = g(G(x))$ —then write  $\phi(x) = G(F^{-1}(x))$  which satisfies  $f(\phi(x)) = \phi(g(x))$ . And then it's required to show  $\phi \in \mathcal{B}$ .

## 2 The First Step

This section will be very quick. It will only require a quick write-up of infinite compositions. We skip to the meat in the following theorem:

**Theorem 2.1.** Let  $f \in \mathcal{B}$ . Let  $p : \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $p'(x)$  exists and  $p'(x) > 0$  for  $x > X$ . Additionally,

$$\sum_{j=1}^{\infty} \|p(x-j)\|_{a \leq x \leq b} < \infty$$

$$\sum_{j=1}^{\infty} \|p'(x-j)\|_{a \leq x \leq b} < \infty$$

for all  $a, b \in \mathbb{R}$ . Then, there exists a function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $\Phi(x+1) = p(x)f(\Phi(x))$ ; and  $\Phi'(x) > 0$  for  $x > X$ . This function is given by,

$$\Phi(x) = \bigotimes_{j=1}^{\infty} p(x-j)f(t) \bullet t$$

*Proof.* To begin, we'll gather our normality theorems. For all compact intervals  $\mathcal{I} \subset \mathbb{R}$ , there exists  $N$  such for  $n, m > N$ ; The term,

$$\left\| \bigotimes_{j=n}^m p(x-j)f(t) \bullet t \right\|_{x \in \mathcal{I}, 0 \leq t \leq 1} < \epsilon$$

Set  $h_j(x, t) = p(x-j)f(t)$  and set  $\|h_j(x, t)\|_{x \in \mathcal{I}, 0 \leq t \leq 1} = \rho_j$ . Pick  $1 > \epsilon > 0$ , and choose  $N$  large enough so when  $n > N$ ,

$$\rho_n < \epsilon$$

Denote:  $\Phi_{nm}(t, x) = \Omega_{j=n}^m h_j(x, t) \bullet x = h_n(x, h_{n+1}(x, \dots, h_m(x, t)))$ . We go by induction on the difference  $m - n = k$ ; which counts how many compositions appear. When  $k = 0$  then,

$$\|\Phi_{nn}(x, t)\|_{\mathcal{I}, 0 \leq t \leq 1} = \|h_n(x, t)\|_{\mathcal{I}, 0 \leq t \leq 1} = \rho_n < \epsilon$$

Assume the result holds for  $m - n < k$ , we show it holds for  $m - n = k$ . Observe,

$$\begin{aligned} \|\Phi_{nm}(x, t)\|_{\mathcal{I}, 0 \leq t \leq 1} &= \|h_n(x, \Phi_{(n+1)m}(x, t))\|_{\mathcal{I}, 0 \leq t \leq 1} \\ &\leq \|h_n(x, t)\|_{\mathcal{I}, 0 \leq t \leq 1} \\ &= \rho_n < \epsilon \end{aligned}$$

Which follows by the induction hypothesis because  $|\Phi_{(n+1)m}(x, t)| < \epsilon < 1$ ; i.e:  $m - n - 1 < k$ . And from this,

$$\left\| \bigotimes_{j=1}^m p(x-j)f(t) \bullet t \right\|_{\mathcal{I}, 0 \leq t \leq 1} < M$$

For some  $M$ . And equally so,

$$\begin{aligned} \left\| \frac{d}{dt} \bigcirc_{j=1}^m p(x-j)f(t) \bullet t \right\|_{\mathcal{I}, 0 \leq t \leq 1} &= \prod_{j=1}^m \left\| p(x-j)f' \left( \bigcirc_{c=j+1}^m p(x-c)f(t) \bullet t \right) \right\|_{\mathcal{I}, 0 \leq x \leq 1} \\ &\rightarrow 0 \text{ as } m \rightarrow \infty \\ &< M \end{aligned}$$

If we choose  $M$  large enough. Call  $\Phi_m(x, t) = \bigcirc_{j=1}^m p(x-j)f(t) \bullet t$ . Now, applying the mean value theorem,

$$\begin{aligned} \|\Phi_{m+1}(x, 0) - \Phi_m(x, 0)\|_{x \in \mathcal{I}} &= \left\| \Phi_m(t, p(x-m-1)f(0)) - \Phi_m(x, 0) \right\|_{x \in \mathcal{I}} \\ &\leq M \|p(x-m-1)\|_{x \in \mathcal{I}} f(0) \end{aligned}$$

And therefore, for all  $\epsilon > 0$  there exists  $N$  such for  $n, m > N$  and for all  $x \in \mathcal{I}$ ,

$$\|\Phi_m(x, 0) - \Phi_n(x, 0)\|_{\mathcal{I}} \leq \sum_{j=n+1}^m M f(0) \|p(x-j)\|_{\mathcal{I}} < \epsilon$$

Which concludes the proof of convergence. This function satisfies the identity  $\Phi(x+1) = p(x)f(\Phi(x))$ . To derive differentiability, we simply describe a formula. Which is,  $\Phi'(x+1) = p'(x)f(\Phi(x)) + p(x)f'(\Phi(x))\Phi'(x)$ . This allows to construct a formula for  $\Phi'$ . Taking the derivative we get,

$$\Phi'(x) = \bigcirc_{j=1}^{\infty} p'(x-j)f(\Phi(x-j)) + p(x-j)f'(\Phi(x-j))t \bullet t$$

Which converges in the exact manner as the above. The function  $\Phi(x) \sim f(0)p(x)$  as  $x \rightarrow -\infty$  which implies  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ . The function  $\Phi'(x) > 0$  by noting this infinite composition, and the fact  $\Phi'(x+1) = p(x)f'(\Phi(x))\Phi'(x) + p'(x)f(\Phi(x))$ ; which has terms  $> 0$  for  $x > X$ .  $\square$

### 3 The Second Step

This section is intended to construct a super-function  $F$  to an arbitrary element  $f \in \mathcal{B}$ . This function should be  $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$  and satisfy the identity  $f(F(x)) = F(x+1)$ . We'll have to split into cases here. The first case is the true non-trivial case, this is when  $f'(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . This equates to when  $x = o(f(x))$ .

We can then write,

$$f(x) = xh(x) \text{ for } X < x < \infty$$

Where,

$$h(x) \rightarrow \infty \text{ as } x \rightarrow \infty$$

This will imply that:  $f^{\circ n}(x) > A^n$  for all  $A > 1$  and  $n, x$  large enough. This is because if  $h$  were constant or were bounded by a constant at infinity then, there always equals an  $A$  such that,  $\frac{f^{\circ n}}{A^n} \rightarrow 0$ . Now the idea is to construct an iteration,

$$\begin{aligned} F(x+1) &= f(F(x)) \\ F(x+1) &= F(x)h(F(x)) \\ F(x+1) &> AF(x) \end{aligned}$$

So we should expect that  $F(x) > e^{Bx}$  for all  $B > 0$ . So when we talk about,

$$\Phi(x) = \prod_{j=1}^{\infty} p(x-j)f(t) \bullet t$$

This function must grow faster than when  $h(t)$  is bounded. And when  $h(t) = H$  we get,

$$\Phi_H(x) = \prod_{j=1}^{\infty} p(x-j)(t+x_0)H \bullet t$$

Which grows like  $e^{x \log p(x) + \log(H)x}$ ; by the relationship  $\Phi_H(x+1) = p(x)H\Phi_H(x)$  as  $x \rightarrow \infty$ . So we should get that  $\Phi(x) \sim e^{x \log(p(x-1)h(\Phi(x-1)))}$  in a close enough way. Because  $\log(h(x)) \rightarrow \infty$  but in the worst case scenario  $\frac{d}{dx} \log(h(x)) \rightarrow 0$ .

At this moment we'd like to make a choice of  $p(x)$ . But what we want from  $p(x)$  is a tad odd and depends on  $f$ . We want a  $p(x)$  such that the following identity holds,

$$p(x)f(\Phi(x)) \leq f(\Phi(x) + e^{\mu x})$$

For some  $\mu > 0$ . Choosing an appropriate  $p$  proves to be the very hard part. But for the moment, assume we have found an appropriate  $p$  which satisfies this identity. We start with the sequence of functions,

$$F_n(x) = f^{\circ -n}(\Phi(x+n))$$

Which, each function  $F_n : \mathbb{R} \rightarrow \mathbb{R}$  and is injective. This will be relatively simple to do thanks to the fact  $f \in \mathcal{B}$ . To begin, the inverse function  $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$  and since  $f'(x) \rightarrow \infty$  as  $x \rightarrow \infty$  we know that,

$$\lim_{x \rightarrow \infty} \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \rightarrow 0$$

Now this implies, for  $x, y > X$  we get the arbitrary Lipschitz constraint,

$$|f^{-1}(x) - f^{-1}(y)| \leq \lambda|x - y|$$

For  $0 < \lambda < 1$ . Now we'll choose  $\lambda$  so that  $\mu$  from our construction of  $p$  satisfies  $e^\mu \lambda < 1$ . And we'll write,

$$F_n(x) = \Phi(x) + \tau_n(x)$$

Where,

$$\tau_{n+1}(x) = f^{-1}(\Phi(x+1) + \tau_n(x+1)) - \Phi(x)$$

And each  $\Phi(x+1) + \tau_n(x) \geq X$ . Now, when we take,

$$|\tau_{n+1}(x) - \tau_n(x)| \leq \frac{1}{f'(f^{-1}(X))} |\tau_n(x+1) - \tau_{n-1}(x+1)|$$

Which we iterate to get,

$$|\tau_{n+1}(x) - \tau_n(x)| \leq \prod_{j=1}^n f'(f^{-1}(X))^{-1} |f^{-1}(\Phi(x+n+1)) - \Phi(x+n)|$$

So long as we remember  $F_n(x) = \Phi(x) + \tau_n(x)$ , where  $\tau_0(x) = 0$  and  $\tau_1(x) = f^{-1}(\Phi(x+1)) - \Phi(x)$ . So long as we keep  $\Phi(x+n) + \tau_n(x+n) > X$  which is always guaranteed for large enough  $x > X'$ . Now, we can derive that,

$$\Phi(x+1) = p(x)f(\Phi(x)) \leq f(\Phi(x) + e^{\mu x})$$

So that  $f^{-1}(\Phi(x+n+1)) = f^{-1}(p(x+n)f(\Phi(x+n))) \leq \Phi(x+n) + e^{\mu(x+n)}$ . Therefore,

$$|\tau_{n+1}(x) - \tau_n(x)| \leq \prod_{j=1}^n f'(f^{-1}(X))^{-1} |e^{\mu(x+n)}|$$

From this, we get that,

$$|F_{n+1}(x) - F_n(x)| \leq \lambda^n e^{\mu n} M$$

For some constant  $M$  depending on a compact interval in  $x > X'$ , and since  $e^\mu \lambda < 1$  we have convergence. The convergence is uniform for compact intervals in  $x > X'$ . This gives us a function,

$$F(x) : \mathbb{R}_{x > X'} \rightarrow \mathbb{R}$$

And since  $f(F_n(x)) = F_{n-1}(x+1)$ , we know that  $f(F(x)) = F(x+1)$ . So by taking repeated  $f^{-1}$ 's we know that,

$$F(x) : \mathbb{R} \rightarrow \mathbb{R}$$

And does so bijectively, because the orbits of  $f^{-1}$  tend to  $-\infty$ ; so that  $F(x - n) \rightarrow -\infty$  as  $n \rightarrow \infty$ . And this nearly takes care of the case when  $f'(x) \rightarrow \infty$ ; given we have an appropriate function  $p$ . We provide such a  $p$  in the following lemma.

**Lemma 3.1.** *Let  $\Phi(x)$  be given by the infinite composition,*

$$\Phi(x) = \bigcirc_{j=1}^{\infty} p(x - j)f(t) \bullet t$$

*There exists an appropriate  $p$  such that,*

$$p(x)f(\Phi(x)) \leq f(\Phi(x) + e^{\mu x})$$

*For  $\mu > 0$ .*

*Proof.* Let's break this proof into cases. First, assume that  $f(x) \geq x^{1+\delta}$  for  $\delta > 0$ . Then,

$$\Phi(x) \geq e^{Ax(\log(x) + \log p(x))}$$

for some  $A > 0$ . Then, let  $p(x) = \mathcal{O}(x^{-\delta/2})$  as  $x \rightarrow \infty$ . Then,

$$x^{-\delta/2}\Phi(x)^{1+\delta} \leq (\Phi(x) + e^{\mu x})^{1+\delta}$$

Then similarly, when we let  $\delta \rightarrow 0$  as  $x \rightarrow \infty$  we have the same consequence. Therefore, if  $\log(h(x)) \geq \delta(x) \log(x)$  where  $\delta(x) \rightarrow 0$  as  $x \rightarrow \infty$  but  $\delta(x) \log(x) \rightarrow \infty$ . Then, choose  $p(x) = \mathcal{O}(x^{-\delta(x)/2})$ . This can be given as,

$$p(x) = \frac{1}{e^{-x} + 1} \frac{1}{(1 + x^2)^{\delta(x)/4}}$$

□

Now when  $f'(x) \rightarrow A$  for  $A > 1$ , then we know that  $f$  has an attracting fixed point at infinity. This means if we take the conjugation  $z \mapsto \frac{1}{z}$  and get a function  $q(x)$ , then  $q(0) = 0$  and  $q'(0) = \frac{1}{A} < 1$ . Therefore there exists a superfunction for  $q$  in a neighborhood  $[0, \epsilon)$ , therein their equals a superfunction  $F$  of  $f$  by undoing the conjugation; for  $F(x) : \mathbb{R}_{x>X} \rightarrow \mathbb{R}$ , and taking iterated  $f^{-1}$ 's extends  $F$  to  $\mathbb{R}$ .

**Theorem 3.2.** *Let  $f \in \mathcal{B}$ . There exists a function  $F : \mathbb{R} \rightarrow \mathbb{R}$  bijectively, where  $F'(x) > 0$  and  $F'(x) \rightarrow \infty$  as  $x \rightarrow \infty$ ; and  $f(F(x)) = F(x + 1)$ .*

The proof of which is above.

## 4 The Third Step

The last step we need in proving the conjugacy property is a tad subtle. Take two elements of  $\mathcal{B}$ , let's say  $f, g$ —and their respective super-functions  $F, G$ . There exists a function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that,

$$\phi(x) = G(F^{-1}(x))$$

And,

$$g(\phi(x)) = g(G(F^{-1}(x))) = G(F^{-1}(x) + 1) = G(F^{-1}(f(x))) = \phi(f(x))$$

And now, we want to show that  $\phi \in \mathcal{B}$ . First of all, it is bijective from  $\mathbb{R} \rightarrow \mathbb{R}$  and is  $\mathcal{C}^1$ . Second of all  $\phi'(x) > 0$ . For the rest of the properties, we have to play very carefully. We know that,

$$\phi'(x) = G'(F^{-1}(x)) \frac{d}{dx} F^{-1}(x)$$

So that,

$$\phi'(F(x)) = \frac{G'(x)}{F'(x)}$$

Upon which, either  $\phi'(x) \rightarrow A > 1$  or  $\phi'(x) \rightarrow \frac{1}{A} < 1$ ; depending on whether  $G'$  is greater than  $F'$ . And then, the inverse function  $\frac{d}{dx} \phi^{-1}(x) \rightarrow A > 1$  or  $\frac{d}{dx} \phi^{-1}(x) \rightarrow \frac{1}{A} < 1$ . This can be related to,

$$g(x) \geq f(x) \Rightarrow G(x) \geq F(x)$$

But, this isn't exactly what we want. We will call,

$$\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = \mathcal{I}_{g,f}$$

Therein, if  $\mathcal{I}_{g,f} > 1$  then  $g > f$ . In such a case, we can define a function  $\phi \in \mathcal{B}$  such that,

$$g(\phi(x)) = \phi(f(x))$$

Given by  $\phi(x) = G(F^{-1}(x))$ ; which only depends on the multiplier at infinity; upon which if it's greater, it allows for a conjugation. The trouble arises when  $\mathcal{I}_{g,f} = 1$ . In this case,  $\phi \in \overline{\mathcal{B}}$  where  $\overline{\mathcal{B}}$  is the closure of  $\mathcal{B}$  as a monoid using the topology of Cauchy convergence on compact intervals of  $\mathbb{R}$ . This is equivalent to allowing  $\bar{f}$  such that  $\lim_{x \rightarrow \infty} \bar{f}'(x) = 1$ .

We solve this problem rather simply. Suppose  $\mathcal{I}_{g,f} = 1$  then let  $g_\mu(x) = g(\mu x)$  for  $\mu > 1$  so that  $\mathcal{I}_{g_\mu, f} > 1$ . Implying there is a function  $\phi_\mu$  such that  $g(\phi_\mu(x)) = \phi_\mu(f(x))$ . Upon which, taking the limit  $\mu \rightarrow 1$  is an exercise left to the reader. This gives the following theorem,

**Theorem 4.1.** *Let  $f, g \in \mathcal{B}$ . Suppose  $\mathcal{I}_{g,f} \geq 1$ . Then there exists a function  $\phi \in \overline{\mathcal{B}}$  such that,*

$$g(\phi(x)) = \phi(f(x))$$

Which concludes this paper.

## 5 Conclusion

This is only a brief sketch of what we can do. It is intended solely as a precursor to a more developed language. It is truly a toy-model, and acts only as such. We thank the reader for taking the time to read this notice.