Solving Conjugacy Equations on $\mathcal{C}^1(\mathbb{R},\mathbb{R})$

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Abstract

The author constructs a space of functions $\mathcal{B} \subset \mathcal{C}^1(\mathbb{R},\mathbb{R}) = \{f(x) \,|\, f : \mathbb{R} \to \mathbb{R},\, f'(x) : \mathbb{R} \to \mathbb{R}\}$ which satisfies the conjugacy property. The conjugacy property can be summarized as $\forall f, g \in \mathcal{B}\ g > f\ \exists \phi \in \overline{\mathcal{B}}$ s.t. $g(\phi(x)) = \phi(f(x))$. This paper is intended as an exercise in functional analysis, and used as an existence method of solutions to function spaces with the conjugacy property.

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1 Introduction

This paper will be a brief notice, and not much more. We intend to construct a manner of designing conjugacy classes under the composition operator. For this reason, we're going to play with toy-models; very simple cases which should help us with more complicated functions. Much of this paper originates from a series of discussions the author had with MphLee on the Tetration forum. This is intended as a proof of concept of much of what we discussed.

For this reason, we're going to work in the space $\mathcal{C}^1(\mathbb{R},\mathbb{R}) = \{f(x) \,|\, f : \mathbb{R} \to \mathbb{R},\, f'(x) : \mathbb{R} \to \mathbb{R}\}$, and pull out a subset of this space which satisfies the conjugacy property. The first thing we'll want is that,

$$f'(x) > 0$$

So that there is an inverse function f^{-1} which is also differentiable. And, to pair hand in hand with this, we also want that,

$$f : \mathbb{R} \to \mathbb{R} \text{ bijectively}$$

Which, can be summarized as $f : \mathbb{R} \to \mathbb{R}$ is bijective with positive derivative. The next things we need from f to construct the space $\mathcal{B}$, we want some slightly different requirements. The first thing we want is that,

$$f'(x) \to A > 1 \text{ as } x \to \infty$$

Where $A \in \mathbb{R} \cup \infty$. So that the derivative grows past 1. This being very important in the manner we construct our conjugate functions; but is less necessary than the above conditions. But still, in the toy model, saves us a bunch of space. The last thing we need, is even more artificially constructed. It is the idea that,

$$f^{\circ -n}(x) \to -\infty$$

For all $x \in (-\infty, \infty)$. This is that the orbits of $f^{-1}(x)$ tend to negative infinity. We need this condition to ensure that $\phi \in \mathcal{B}$. If we were to drop this condition, we could still construct $f(\phi(x)) = \phi(g(x))$ but it is no longer necessary that $\phi \in \mathcal{B}$.

We will call this our set $\mathcal{B}$. Which is given as,

$$\begin{aligned}\mathcal{B} \;=\; & \{f(x)\,|\,f \in \mathcal{C}^1(\mathbb{R},\mathbb{R}),\, f'(x) > 0,\, f \text{ is a bijection},\, f'(x) \to A > 1 \,\mathrm{as}\, x \to \infty, \\ & f^{\circ -n}(x) \to -\infty \,\mathrm{as}\, n \to \infty\}\end{aligned}$$

This set $\mathcal{B}$ is a monoid under composition. This is to mean, for $f, g \in \mathcal{B}$ we have $f(g) \in \mathcal{B}$. Which the only thing that is non-trivial is that the orbits tend to negative infinity, but it's apparent because $f^{-1}(x) < x$ eventually, and so must $g^{-1}(f^{-1}(x)) < x$, and so the result happens similarly.

Therein, the main result of this notice, is to show that $\mathcal{B}$ satisfies the conjugacy property. Which is,

$$\forall f, g \in \mathcal{B} :\; \exists \phi \in \overline{\mathcal{B}}\; f(\phi(x)) = \phi(g(x)) \text{ or } \exists \phi \in \overline{\mathcal{B}}\; g(\phi(x)) = \phi(f(x))$$

Where $\overline{\mathcal{B}}$ is the closure of $\mathcal{B}$ to include $\lim_{x\to\infty} f'(x) = 1$. The manner we do this is in a chain of constructions. We begin by constructing, $\Phi(x) : \mathbb{R} \to \mathbb{R}^+$; such that $\Phi(x+1) = e^x f(\Phi(x))$. The second construction is to use Φ to construct $F : \mathbb{R} \to \mathbb{R}$ such that $F(x+1) = f(F(x))$. And then, constructing in the same manner a function G such that $G(x+1) = g(G(x))$–then write $\phi(x) = G(F^{-1}(x))$ which satisfies $f(\phi(x)) = \phi(g(x))$. And then it's required to show $\phi \in \mathcal{B}$.

2 The First Step

This section will be very quick. It will only require a quick write-up of infinite compositions. We skip to the meat in the following theorem:

Theorem 2.1. *Let $f \in \mathcal{B}$. There exists a function $\Phi : \mathbb{R} \to \mathbb{R}^+$ such that $\Phi'(x) > 0$ and $\Phi'(x) \to \infty$ as $x \to \infty$; such that $\Phi(x+1) = e^x f(\Phi(x))$. This function is given by,*

$$\Phi(x) = \underset{j=1}{\overset{\infty}{\Omega}} e^{x-j} f(t) \bullet t$$

Proof. To begin, we'll gather our normality theorems. For all compact intervals $\mathcal{I} \subset \mathbb{R}$, there exists N such for $n, m > N$; The term,

$$\Big|\Big| \underset{j=n}{\overset{m}{\Omega}} e^{x-j} f(t) \bullet t \Big|\Big|_{x \in \mathcal{I}, 0 \leq t \leq 1} < \epsilon$$

Set $h_j(x,t) = e^{x-j} f(t)$ and set $||h_j(x,t)||_{x \in \mathcal{I}, 0 \leq t \leq 1} = \rho_j$. Pick $1 > \epsilon > 0$, and choose N large enough so when $n > N$,

$$\rho_n < \epsilon$$

Denote: $\Phi_{nm}(t,x) = \Omega_{j=n}^m h_j(x,t) \bullet x = h_n(x, h_{n+1}(x, ...h_m(x,t)))$. We go by induction on the difference $m - n = k$; which counts how many compositions appear. When $k = 0$ then,

$$||\Phi_{nn}(x,t)||_{\mathcal{I}, 0 \leq t \leq 1} = ||h_n(x,t)||_{\mathcal{I}, 0 \leq t \leq 1} = \rho_n < \epsilon$$

Assume the result holds for $m - n < k$, we show it holds for $m - n = k$. Observe,

$$\begin{aligned} ||\Phi_{nm}(x,t)||_{\mathcal{I}, 0 \leq t \leq 1} &= ||h_n(x, \Phi_{(n+1)m}(x,t))||_{\mathcal{I}, 0 \leq t \leq 1} \\ &\leq ||h_n(x,t)||_{\mathcal{I}, 0 \leq t \leq 1} \\ &= \rho_n < \epsilon \end{aligned}$$

Which follows by the induction hypothesis because $|\Phi_{(n+1)m}(x,t)| < \epsilon < 1$; i.e: $m - n - 1 < k$. And from this,

$$|| \underset{j=1}{\overset{m}{\Omega}} e^{x-j} f(t) \bullet t ||_{\mathcal{I}, 0 \leq t \leq 1} < M$$

For some M. And equally so,

$$\begin{aligned} ||\frac{d}{dt} \underset{j=1}{\overset{m}{\Omega}} e^{x-j} f(t) \bullet t||_{\mathcal{I}, 0 \leq t \leq 1} &= \prod_{j=1}^{m} \Big|\Big| e^{x-j} f'(\underset{c=j+1}{\overset{m}{\Omega}} e^{x-c} f(t) \bullet t) \Big|\Big|_{\mathcal{I}, 0 \leq x \leq 1} \\ &\to 0 \text{ as } m \to \infty \\ &< M \end{aligned}$$

If we choose M large enough. Call $\Phi_m(x,t) = \Omega_{j=1}^m e^{x-j} f(t) \bullet t$. Now, applying the mean value theorem,

$$\begin{aligned} ||\Phi_{m+1}(x,0) - \Phi_m(x,0)||_{t\in\mathcal{I}} &= \Big|\Big|\Phi_m(t, e^{x-m-1}f(0)) - \Phi_m(x,0)\Big|\Big|_{t\in\mathcal{I}} \\ &\leq Me^{x-m-1}f(0) \end{aligned}$$

Which concludes the proof of convergence. This function satisfies the identity $\Phi(x+1) = e^x f(\Phi(x))$. To derive differentiability, we simply describe a formula. Which is, $\Phi'(x+1) = e^x(f(\Phi(x)) + f'(\Phi(x))\Phi'(x))$. This allows to construct a formula for Φ'. Taking the derivative we get,

$$\Phi'(x) = \mathop{\Omega}_{j=1}^{\infty} e^{x-j}(f(\Phi(x-j)) + f'(\Phi(x-j))t) \bullet t$$

Which converges in the exact manner as the above. It is trivial to show $\Phi'(x) > 0$ and $\Phi'(x) \to \infty$ as $x \to \infty$. And $\Phi(x) \sim f(0)e^x$ as $x \to -\infty$ which implies $\Phi : \mathbb{R} \to \mathbb{R}^+$. $\square$

3 The Second Step

This section is intended to construct a super-function F to an arbitrary element $f \in \mathcal{B}$. This function should be $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ and satisfy the identity $f(F(x)) = F(x+1)$. We start with the sequence of functions,

$$F_n(x) = f^{\circ -n}(\Phi(x+n))$$

Which, each function $F_n : \mathbb{R} \to \mathbb{R}$ and is injective. This will be relatively simple to do thanks to the fact $f \in \mathcal{B}$. To begin, the inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ and since $f'(x) \to A > 1$ as $x \to \infty$ we know that,

$$\lim_{x\to\infty} \frac{d}{dx} f^{-1}(x) \to \frac{1}{A}$$

Now this implies, for $x > X$ we get the Lipschitz constraint,

$$|f^{-1}(x) - f^{-1}(y)| \leq \lambda |x-y|$$

For $0 < \frac{1}{A} < \lambda < 1$. Now, when we take,

$$|F_{n+1}(x) - F_n(x)| \leq \lambda |F_n(x+1) - F_{n-1}(x+1)|$$

Which we iterate to get,

$$|F_{n+1}(x) - F_n(x)| \leq \lambda^n |x+n|$$

So long as we keep $\Phi(x+n) > X$ which is always guaranteed for large enough $x > X'$ and because $f^{-1}F(x+1) > F(x)$. This gives us a function,

$$F(x) : \mathbb{R}_{x>X'} \to \mathbb{R}$$

And by taking repeated f^{-1}'s we know that,

$$F(x) : \mathbb{R} \to \mathbb{R}$$

And does so bijectively, because the orbits of f^{-1} tend to $-\infty$; so that $F(x-n) \to -\infty$ as $n \to \infty$.

Theorem 3.1. *Let $f \in \mathcal{B}$. There exists a function $F : \mathbb{R} \to \mathbb{R}$ bijectively, where $F'(x) > 0$ and $F'(x) \to \infty$ as $x \to \infty$. This function F is given as,*

$$\lim_{n\to\infty} f^{\circ -n}(\Phi(x+n)) = F(x)$$

The proof of which is above.

4 The Third Step

The last step we need in proving the conjugacy property is a tad subtle. Take two elements of $\mathcal{B}$, let's say f, g–and their respective super-functions F, G. There exists a function $\phi : \mathbb{R} \to \mathbb{R}$ such that,

$$\phi(x) = G(F^{-1}(x))$$

And,

$$g(\phi(x)) = g(G(F^{-1}(x))) = G(F^{-1}(x) + 1) = G(F^{-1}(f(x))) = \phi(f(x))$$

And now, we want to show that $\phi \in \mathcal{B}$. First of all, it is bijective from $\mathbb{R} \to \mathbb{R}$ and is $\mathcal{C}^1$. Second of all $\phi'(x) > 0$. For the rest of the properties, we have to play very carefully. We know that,

$$\phi'(x) = G'(F^{-1}(x))\frac{d}{dx}F^{-1}(x)$$

So that,

$$\phi'(F(x)) = \frac{G'(x)}{F'(x)}$$

Upon which, either $\phi'(x) \to A > 1$ or $\phi'(x) \to \frac{1}{A} < 1$; depending on whether G' is greater than F'. And then, the inverse function $\frac{d}{dx}\phi^{-1}(x) \to A > 1$ or $\frac{d}{dx}\phi^{-1}(x) \to \frac{1}{A} < 1$. This can be related to,

$$g(x) \geq f(x) \;\Rightarrow\; G(x) \geq F(x)$$

But, this isn't exactly what we want. We will call,

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \mathcal{I}_{g,f}$$

Therein, if $\mathcal{I}_{g,f} > 1$ then $g > f$. In such a case, we can define a function $\phi \in \mathcal{B}$ such that,

$$g(\phi(x)) = \phi(f(x))$$

Given by $\phi(x) = G(F^{-1}(x))$; which only depends on the multiplier at infinity; upon which if it's greater, it allows for a conjugation. The trouble arises when $\mathcal{I}_{g,f} = 1$. In this case, $\phi \in \overline{\mathcal{B}}$ where $\overline{\mathcal{B}}$ is the closure of $\mathcal{B}$ as a monoid. This is equivalent to allowing $\overline{f}$ such that $\lim_{x\to\infty} \overline{f}'(x) = 1$.

We solve this problem rather simply. Suppose $\mathcal{I}_{g,f} = 1$ then let $g_\mu(x) = g(\mu x)$ for $\mu > 1$ so that $\mathcal{I}_{g_\mu,f} > 1$. Implying there is a function ϕ_μ such that $g(\phi_\mu(x)) = \phi_\mu(f(x))$. Upon which, taking the limit $\mu \to 1$ is an exercise left to the reader. This gives the following theorem,

Theorem 4.1. *Let $f, g \in \mathcal{B}$. Suppose $\mathcal{I}_{g,f} \geq 1$. Then there exists a function $\phi \in \overline{\mathcal{B}}$ such that,*

$$g(\phi(x)) = \phi(f(x))$$

Which concludes this paper.

5 Conclusion

This is only a brief sketch of what we can do. It is intended solely as a precursor to a more developed language. It is truly a toy-model, and acts only as such. We thank the reader for taking the time to read this notice.